POLYNOMIAL IDENTITIES FOR 2 x 2 MATRICES*)

Vesselin Drensky
Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria

0. Introduction

Let \( K\langle X \rangle \) be the free associative algebra freely generated by a countable set of symbols \( X = \{x_1, x_2, \ldots \} \) over a field \( K \) of characteristic 0. The element \( f(x_1, \ldots, x_m) \in K\langle X \rangle \) is called a polynomial identity for a \( K \)-algebra \( R \) if \( f(r_1, \ldots, r_m) = 0 \) for all \( r_1, \ldots, r_m \in R \). The set

\[ T(R) = \{ f(x_1, \ldots, x_m) \in K\langle X \rangle \mid f \text{ is a polynomial identity for } R \} \]

is a two-sided ideal of \( K\langle X \rangle \) which is invariant under all endomorphisms of \( K\langle X \rangle \) and is called a T-ideal. When \( T(R) \neq 0 \), i.e. \( R \) satisfies a non-trivial polynomial identity, \( R \) is a PI-algebra. The class \( \operatorname{var} R \) of all algebras satisfying the identities from \( T(R) \) is called the variety of algebras generated by \( R \).

In the modern treatment of PI-algebras a basic problem is to describe quantitatively the polynomial identities of an algebra \( R \). The natural measures for \( T(R) \) are the multilinear codimension sequence \( c_n(R), n = 1, 2, \ldots, \) and the Hilbert (or Poincaré) series \( \mathcal{H}(F_m(\operatorname{var} R), t_1, \ldots, t_m) \) of the relatively free algebra of rank \( m \) (or universal PI-algebra [20])

\[ F_m(\operatorname{var} R) = K\langle x_1, \ldots, x_m \rangle / (K\langle x_1, \ldots, x_m \rangle \cap T(R)) \]

in the variety \( \operatorname{var} R \). More precisely \( c_n(R) \) is the degree of

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the $S_n$-character $\chi_n(R)$ canonically related with $T(R)$ and $H(F_m(\text{var } R), t_1, \ldots, t_m)$ is the character of the $GL_m$-module $F_m(\text{var } R)$. Many results, both old and very recent, give explicit values or estimates for $c_n(R)$, $\chi_n(R)$ and $H(F_m(\text{var } R), t_1, \ldots, t_m)$.

Especially attractive is the quantitative theory of the polynomial identities of the algebra $M_k$ of the $k \times k$ matrices. Since the algebra $F_m(\text{var } M_k)$ is isomorphic to the algebra generated by $m$ generic $k \times k$ matrices, in the study of $T(M_k)$ the invariant theory of matrices [14, 17] was successfully involved. We refer to [8] for a survey on the results before 1982.

Up till now the results on $T(M_k)$, $k > 2$, are far from their final form and only the structure of the identities of the $2 \times 2$ matrices is well known. In a series of papers (see [19] for a survey) Regev obtained the asymptotic behaviour of $c_n(M_2)$ and $\chi_n(M_2)$; the explicit expression of $\chi_n(M_2)$ was found by Formanek [9] and the author [5]; Procesi [15] computed the codimension sequence of $M_2$. Formanek, Halpin and Li [10] for $m = 2$ and Procesi [15], Formanek [9] and the author [5] in the general case computed the Hilbert series of $F_m(\text{var } M_2)$.

One of the possible ways to study $T(M_2)$ is via the generic trace algebra generated by $m$ generic $2 \times 2$ matrices together with all the traces of the generic matrix algebra $F_m(\text{var } M_2)$ and the approach of Regev, Formanek and Procesi followed this way. For a survey on the $2 \times 2$ generic trace algebra see [12].

The purpose of the present paper is to give a revised version