Group invariance of a neo-Hookean system: incorporation of stretch change

D. LEVI
Università di Roma, Roma, Italia

C. ROGERS*
University of Waterloo, Waterloo, Canada

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Abstract. Invariance of the nonlinear elasto-static system descriptive of the plane-strain deformation of a neo-Hookean material is determined under Lie group transformations which accommodate change in stretch. The associated finite transformations are constructed and the results are set in the context of previous work in the area.

1. Introduction

Lie group methods have been applied in nonlinear elastostatics notably by Hill [1, 2]. Furthermore, in [3] Abeyaratne and Horgan obtained an exact solution for the axisymmetric deformation and stress distribution associated with a boundary value problem involving the deformation of an infinite medium composed of Blatz-Ko material containing a circular cylindrical cavity under pressure loading conditions. In a subsequent paper [4], exact solutions to boundary value problems describing the finite elastostatic deformation of hollow circular cylinders and spheres of Blatz-Ko material under internal pressure were obtained. The analytic progress made in these interesting works on Blatz-Ko media turns upon a reduction of the nonlinear equation governing the radial displacement which may be obtained by use of group invariance to introduce canonical coordinates (Rogers [5]). On the other hand, in finite elasto-dynamics the nonlinear superposition principle for Pinney’s equation employed by Shahinpoor and Nowinski [6] in their analysis of the large amplitude oscillation of thin shells of Mooney-Rivlin material is a consequence of group invariance (Rogers and Ames [7]).

Hill [2] noted that the important solution of Ericksen’s problem originally due to Klingbeil and Shield [8] and Singh and Pipkin [9] may be readily
retrieved via group methods. In this note, the group invariance obtained by Hill is shown to be embedded in a broader class of Lie transformations which link deformations with different stretches. An invariance result due to Adkins [10] is obtained corresponding to a particular subgroup.

2. The nonlinear elastostatic system. Group invariance

With material and spatial rectangular cartesian coordinates \((X, Y, Z)\) and \((x, y, z)\) respectively, the nonlinear system governing the plane strain deformation with constant stretch \(\lambda\)

\[
X = F(x, y), \quad Y = G(x, y), \quad Z = \lambda^{-1}z, \tag{2.1}
\]

of a neo-Hookean material consists of (Truesdell and Noll [11])

\[
\partial(F, G) \partial F \partial G \partial x \partial y - \partial F \partial G \partial y \partial x = \lambda, \tag{2.2}
\]

together with

\[
\frac{\partial(\nabla^2 F, F)}{\partial(x, y)} + \frac{\partial(\nabla^2 G, G)}{\partial(x, y)} = 0 \tag{2.3}
\]

where \(\nabla^2\) is the two-dimensional Laplacian given by

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.4}
\]

It is noted that (2.2) results from the assumption of material incompressibility while (2.3) is a consequence of the neo-Hookean stress-strain relations and the equilibrium equations.

It is known that the system (2.2)–(2.3) is invariant under the inverse transformation

\[
x' = F, \quad y' = G,
\]

\[
F' = x, \quad G' = y\tag{2.5}
\]

\[
\lambda' = \frac{1}{\lambda}.
\]