(Φ, Δ)-TYPE PROBABILISTIC CONTRACTOR AND SOLUTIONS FOR A
NONLINEAR OPERATOR EQUATIONS IN MENER PN-SPACES*

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Abstract

The purpose of this paper is to introduce the concept of (Φ, Δ)-type probabilistic
contractor in Menger PN-spaces and to study the existence and uniqueness of
solutions for the nonlinear operator equations with such probabilistic contractor in
Menger PN-spaces. The results presented in this paper improve and extend the
corresponding results in [1] and [4–8].

Key words Menger PN-space, t-norm of h-type, probabilistic contractor,
operator equation

I. Introduction and Definitions

Recently, Zhang Shisheng† introduced the concept of probabilistic contractor in
probabilistic normed spaces (for short PN-spaces), which is defined by means of a function
φ(t), and proved the existence and uniqueness theorems of solutions for nonlinear operator
equations with the probabilistic contractor in non-Archimedean Menger PN-spaces with a
t-norm of h-type. Later, Zhang Shisheng and Peng Yongcheng† also studied the existence and
uniqueness of solutions for a system of nonlinear operator equations with probabilistic
contractor couples in this kind of spaces.

It is well-known that the non-Archimedean Menger PN-spaces are a special subclass of
Menger PN-spaces. This leads naturally to the following question: Can we establish the
existence and uniqueness theorems of solutions for nonlinear operator equations with the
probabilistic contractor in the general Menger PN-spaces?

In [1] and [4], a special case φ(t) = t/M (0 < M < 1) is investigated. In the present paper,
we shall go further into this question. We introduce the concept of (Φ, Δ)-type probabilistic
contractor, and establish the existence and uniqueness theorems of solutions for nonlinear
operator equations with such probabilistic contractor in the general Menger PN-spaces with a
t-norm of h-type. As an application, we prove two fixed point theorems in Menger PN-spaces
and normed spaces. Our results improve and generalize the corresponding results of [1] and
[4–8].

Throughout this paper let \( \mathbb{R} = (-\infty, +\infty) \), \( \mathbb{R}^+ = [0, +\infty) \), \( \mathbb{Z}^+ \) be the set of all
positive integers. We denote by \( \mathcal{D} \) the set of all left-continuous distribution functions and

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denote by $H$ the specific distribution function defined by
\[
H(t) = \begin{cases} 
1, & t > 0 \\
0, & t \leq 0 
\end{cases}
\]

For the definition and related terminologies of Menger PN-spaces we refer to [3].

**Definition 1**[3] The $t$-norm is called of $h$-type, if the family of functions \{$\Delta_i(t)$\} is equicontinuous at $t = t$, where
\[
\Delta_i(t) = \Delta(t,h_{\Delta}^{-1}(t)), \quad t \in [0,1], \quad m = 1,2,\ldots
\]

**Remark 1** $\Delta = \min$ is a $t$-norm of $h$-type. An other example of $t$-norm of $h$-type is given in [7].

**Definition 2** The function $\Phi: \mathbb{R} \rightarrow \mathbb{R}^+$ is said to satisfy the condition (\(\Phi\)), if $\Phi(t)$ is nondecreasing and the series $\sum_{n=1}^{\infty} \Phi^n(t)$ is convergent for any $t > 0$, where $\Phi^n(t)$ denotes the $n$-th iteration of $\Phi(t)$.

It is easy to see that if the function $\Phi: \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies the condition (\(\Phi\)), then
\[
\lim_{n \rightarrow \infty} \Phi^n(t) = 0, \quad \Phi(0) = 0 \quad \text{and} \quad \Phi(t) < t \quad (\forall t > 0).
\]

**Definition 3** Let $X$, $Y$ be two linear spaces. A mapping $T: Y \rightarrow X$ is said to be odd, if $T(-y) = -T(y)$ ($\forall y \in Y$).

In the following, we denote by $L(Y, X)$ the set of all odd mappings from $Y$ into $X$.

**Definition 4** Let $(X, \mathcal{F}, A)$ and $(Y, \tilde{\mathcal{F}}, A)$ be two Menger PN-spaces and $P: D(P) \subset X \rightarrow Y$, $\Gamma: X \rightarrow L(Y, X)$, $\Gamma$ is called a $(\Phi, A)$-type probabilistic contractor of $P$, if there exists a function $\Phi: \mathbb{R} \rightarrow \mathbb{R}^+$ satisfying the condition (\(\Phi\)) such that for any $x \in D(P)$ and $y \in Y$ the following hold: $x + \Gamma(x)y \in D(P)$ and
\[
\Pi_{\Gamma(y)P(x)}(\Phi(t)) \geq \Delta(\hat{\Pi}_x(t), \Delta(\hat{\Pi}_{\Gamma(y)P(x)}(t), \hat{\Pi}_{\Gamma(y)P(x)}(t))), \quad (\forall t \geq 0) \quad (1.1)
\]

**Main Results**

**Theorem 1** Let $(X, \mathcal{F}, A)$ and $(Y, \tilde{\mathcal{F}}, A)$ be two Menger PN-spaces with a $t$-norm $\Delta$ of $h$-type, and $(X, \mathcal{F}, A)$ $\mathcal{F}$-complete. Let $P: D(P) \subset X \rightarrow Y$ be a $\mathcal{F}$-closed operator and $\Gamma: X \rightarrow L(Y, X)$. If the following conditions are satisfied:

(1) $\Gamma$ is a $(\Phi, A)$-type probabilistic contractor of $P$;

(2) There exists a constant $M > 0$ such that for any $x \in D(P)$ and $y \in Y$
\[
F_{\Gamma(x)P(y)}(t) \geq \hat{F}_\pi(t/M), \quad (\forall t \geq 0) \quad (2.1)
\]

Then the following nonlinear operator equation
\[
P_x = \theta \quad (2.2)
\]
has a solution in $D(P)$ and for any given $x_0 \in D(P)$, the iterative sequence
\[
x_n = x_n - \Gamma(x_n) (P x_n) \quad (2.3)
\]
converges to a solution of (2.2).

Especially, if there exists some $\hat{x} \in X$ such that $\Gamma(\hat{x}): Y \rightarrow X$ is a surjection, then Eq. (2.2) has a unique solution in $D(P)$.

**Proof** It follows from the condition (1) and (2.3) that $\{x_n\} \subset D(P)$ and