ON THE THEORY OF THE WELL-MIXED LAYER CONTAINING A ZERO-ORDER JUMP

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Abstract. Within the framework of a mixed layer (ML) containing a zero-order jump, the concept of ML is generalized for the case of horizontal non-homogeneity on the assumption that not only potential temperature, but also the wind does not change with height. It turns out that the components of the vertical turbulent stresses are quadratic functions of height.

For such a well-mixed layer (WML), bounded below by uneven terrain with an adjacent surface layer, and above – by a stably stratified quasigeostrophic baroclinic atmosphere, a consistent system of equations with all terms independent of height, is obtained. This can be considered as a meteorological generalization of the known shallow-water equations.

As an example of the use of these equations, an analytical solution of the large scale one dimensional steady-state problem concerning the development of the WML in a stable stratified barotropic air mass moving over a heated horizontal surface has been found.

1. Introduction

Mixed-layer models with zero-order jumps are not very complicated from a mathematical point of view; yet they provide a reasonably good description of the phenomena. Therefore they may, perhaps, serve as convenient tools to facilitate the solution of some very complicated problems involving the atmospheric boundary layer (for example, three-dimensional problems concerning air and pollution motion over non-homogeneous terrain: complex problems with several interacting systems, e.g., the soil, the vegetation cover and planetary boundary layer (PBL) of the atmosphere).

In these mixed-layer models, relationships at a surface of discontinuity overlying the mixed layer play an important role. One such relationship (Lilly, 1968) was derived mathematically for the case of the horizontally homogeneous mixed layer. Berkofsky (1982) showed that Lilly's derivation can be generalized for the non-homogeneous case and he formulated a more general layer problem for subsequent numerical solution.

In the present paper, we consider these relationships in more general form in order to obtain a consistent system of equations for the well-mixed layer with zero-order jump. As an example of the application of these equations, we then consider an idealized two-dimensional steady-state problem concerning the development of the well-mixed layer in a stably stratified air mass moving over heated terrain.
2. Relationships at a Surface of Discontinuity

We proceed from the general PBL equations, which can, with sufficient accuracy, be presented in the form*:

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u^2 + \frac{\partial}{\partial y} uv + \frac{\partial}{\partial z} uw = - \frac{\partial \pi}{\partial x} + fv + \frac{\partial \tau_x}{\partial z} \]  
\[ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} uv + \frac{\partial}{\partial y} v^2 + \frac{\partial}{\partial z} vw = - \frac{\partial \pi}{\partial y} - fu + \frac{\partial \tau_y}{\partial z} \]  
\[ 0 = -\frac{\theta}{\theta_0} \frac{\partial \pi}{\partial z} - g \]  
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  
\[ \frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x} u\theta + \frac{\partial}{\partial y} v\theta + \frac{\partial}{\partial z} w\theta = - \frac{\partial H}{\partial z} . \]

Here \( \tau_x = -(u' w') \), \( \tau_y = -(v' w') \), \( H = \overline{\theta' w'} \), \( u, v, w, \theta \) are ensemble-mean values of the three velocity components and potential temperature, respectively, and \( f, g, \theta_0 \) are constants. A definition of \( \theta_0 \) will be given later. The rest of the notation is conventional. In this system, the continuity Equation (4) does not allow for compressibility. Such a simplification is justified for many problems concerning the atmospheric boundary layer. Furthermore, for convenience of transformations, the pressure gradients are expressed in terms of the potential temperature \( \theta \) and a new variable

\[ \pi = C_p \theta_0 \left( \frac{p}{1000 \text{ mb}} \right)^{R/C_p} . \]

We define a surface

\[ z = \delta_z = \delta + z_0 + h, \]

as a boundary between the surface layer and the boundary layer. Here \( \delta(x, y) \) is the height of the terrain, whose slope is sufficiently small so that the hydrostatic equation is satisfied, \( h \) is the surface layer thickness, and \( Z_0 \) is roughness length. They are assumed to be known and constant. At this boundary, the following evident relations have to be satisfied.

\[ \tau_x = C_D u\mu , \quad \tau_y = C_D v\mu \]
\[ H = C_H (\theta - \theta_0) \mu (\mu = \sqrt{u^2 + v^2}) \]

at \( z = \delta_z \).

We assume that the transfer coefficients \( C_D, C_H \) are constant, and that the potential temperature \( \theta_0 \) at the earth’s surface is a given function of \( x, y, t \).

* It is easy to show that the coefficient \( \partial \pi/\partial x \) and \( \partial \pi/\partial y \) can be replaced by unity without loss of accuracy of Equations (1) and (2).