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Categorical Aspects of Equivariant Homotopy

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Abstract. We use the language of homotopy coherent ends and coends, and of homotopy coherent Kan extensions, to give enriched versions of results of Elmendorff. This enables a description of the homotopy type of the space of maps between two G-complexes to be given.

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1. Introduction

Recently we finished working on an article [9] that lays the foundations of an infinitely lax or homotopy coherent version of category theory based on simplicially enriched categories. (That article is the overdue extended version of the preprints, [4] and [7].) The history of homotopy coherence is linked strongly to that of equivariant homotopy theory through the work of Elmendorf, May, Dwyer and Kan, and others. In this paper we will use equivariant homotopy theory as a case study for how our new methods relate to old ones, and in the process will extend results of Dwyer and Kan to an enriched setting. We will try to illustrate the uses and interpretation of the categorical results that we have proved, as we believe that these results will have significant applications in other parts of the subject.

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The diagrams in this article were produced using Paul Taylor's "Diagrams" package. The authors would like to thank him for the use of these macros.

2. G-Sets and OrG-Diagrams

The results in this section are well known, but usually are not presented in this way. We do so here to try to explain why various constructions work in later sections. For simplicity, we restrict attention in this section to G being a discrete group.

We will denote by OrG the orbit category of G. This has as objects the various G-sets G/H, H a subgroup of G, with a morphism from G/H to G/K being a G-equivariant map of G-sets. As $G^{\mathrm{op}} \cong OrG(G/1, G/1)$, there is a functor

$$\phi\colon G^{\mathrm{op}} \longrightarrow OrG,$$

where on the left G is considered as a category with one object, $*$, say. The category of left G-sets is the category $Sets^G$ and the category of OrG^{op}-diagrams is $Sets^{OrG^{\mathrm{op}}}$. These are linked by various functors:

- $\phi^* = Sets^{\phi^{\mathrm{op}}}\colon Sets^{OrG^{\mathrm{op}}} \to Sets^G$, defined by composition along ϕ. This sends an OrG^{op}-diagram Y to $Y(G/1)$, which is a left G-set.
- The right Kan extension along ϕ gives a right adjoint to ϕ^*. This will be denoted here by $R_\phi\colon Sets^G \to Sets^{OrG^{\mathrm{op}}}$. (Later we will tend to write simply R.) It is given by the end formula

  $$R_\phi(X)(G/H) = \int_* Sets(OrG^{\mathrm{op}}(G/H, G/1), X(*))$$

  which is isomorphic to $X^H = \{x\colon\ x \in X,\ h.x = x \text{ for all } h \in H\}$. Right adjointness of R_ϕ is a consequence of its construction as a right Kan extension, but is easily checked directly. The functor R_ϕ, or rather its analogue in other situations, is the key to translating G-equivariant homotopy theory to a 'diagrammatic' form which is more amenable to analysis.
- The functor R_ϕ has in its turn a right adjoint, which will be denoted by $c\colon Sets^{OrG^{\mathrm{op}}} \to Sets^G$. This can be calculated as a left Kan extension, so that $c(Y) = Lan_{R_\phi} Id(Y)$. This gives a coend formula:

  $$c(Y) = \int^X Sets^{OrG^{\mathrm{op}}}(R_\phi(X), Y)\overline{\otimes} X,$$

  where $A\overline{\otimes}B$ is the A-fold copower of B.

Given the construction of coends in $Sets^G$, this object is given by

$$c(Y) = \coprod_X Sets^{OrG^{\mathrm{op}}}(R_\phi(X), Y)\overline{\otimes} X)/\sim$$

where $f'R_\phi(\alpha)\overline{\otimes} x \sim f'\overline{\otimes}\alpha(x)$, where $\alpha\colon X \to X'$, $f'\colon R_\phi(X') \to Y$. This construction as it now stands is 'illegal' as its disjoint union is indexed by a