Dual Estimates in Multiextremal Problems

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Abstract. We propose a technique of improving the dual estimates in nonconvex multiextremal problems of mathematical programming, by adding some additional constraints which are the consequences of the original constraints. This technique is used for the problems of finding the global minimum of polynomial functions, and extremal quadratic and boolean quadratic problems. In the article one ecological multiextremal problem and an algorithm for finding the dual estimate for it also considered. This algorithm is based upon a scheme of decomposition and nonsmooth optimization methods.

Key words. Dual estimates, polynomial functions, non-convex quadratic problems, nondifferentiable optimization.

In this report the general mathematical programming problem find:

\[ f^* = \inf_{x \in X} f_0(x), \quad X \subseteq E^n \]

subject to the constraints:

\[ f_i(x) \leq 0, \quad i \in I_1; \quad f_i(x) = 0, \quad i \in I_2 \]

is considered.

Let \( u \) be a vector of Lagrange multipliers and

\[ L(x, u) = f_0(x) + \sum_{i \in I_1 \cup I_2} u_i f_i(x) \]

be the Lagrange function.

On the set \( U = \{ u : u_i \geq 0, i \in I_1 \} \) let us consider the function:

\[ \Psi(u) = \inf_{x \in X} L(x, u). \]

The value \( \Psi^* = \sup_{u \in U} \Psi(u) \) is called the dual estimate for \( f^* \). It is clear that for \( u \in U \) \( \Psi(u) \leq f^* \) and consequently \( \Psi^* \leq f^* \); therefore, this dual estimate is a lower estimate for \( f^* \). In the nonconvex case the so-called "estimation gap" may occur:

\[ \Delta := f^* - \Psi^* > 0. \]
One of the ways to diminish this gap consists in adding to the constraints in (2) formally new constraints which are consequences of the constraints (2). So, the set of feasible points of the problem (1)–(2) is not changed, but the set of Lagrange variables is extended. In some cases the gap can be reduced to zero by this way. In this article some examples are considered to illustrate such an approach.

1. The Global Minimization Problem for a Polynom

Let a (bounded-from-below) polynomial \( P(x_1, x_2, \ldots, x_n) \) be given and let \( P^* \) be the value of the polynomial at the global minimum point.

By introducing new variables and making use of quadratic substitutions of the form: \( x_i^2 = y_i; x_i x_k = z_{ik} \) and so forth we can reduce the minimization problem for the polynomial \( P(x_1, \ldots, x_n) \) to a quadratic extremal problem with constraints in form of equalities. The direct application of the dual estimate technique to this quadratic problem results in nontrivial estimates only in rare cases. But if we modify the quadratic problem by generating simple quadratic equalities of the quadratic problem variables and by adding these equalities to the constraints some interesting results can be obtained for the modified problem.

THEOREM 1. The dual estimate for the modified quadratic problem, which is equivalent to the problem of minimization of the polynomial \( P(x) = P(x_1, x_2, \ldots, x_n) \) equals \( P^* \) iff the nonnegative polynomial \( P(x_1, x_2, \ldots, x_n) := P(x_1, x_2, \ldots, x_n) - P^* \) can be represented as the sum of squares of other polynomials. Particularly, for \( n = 1 \) the dual estimate is exact.

First, we give a more precise definition of the “modified quadratic problem” in Theorem 1 and an illustrative example.

Let \( P^* > -\infty \). Then the highest degree \( S_i \) of each of the variables \( x_i \) must be even. Let \( S_i = 2 \alpha_i, i = 1, \ldots, n \). Consider integer vectors \( a = \{a_1, \ldots, a_n\} \) with non-negative elements and monomials of the type:

\[
R[a] = x_1^{a_1} \cdots x_n^{a_n}, \quad 0 \leq a_i \leq 1, \quad i = 1, \ldots, n. \tag{3}
\]

Then one obtains a system of identity relations:

\[
R[a^{(1)}]R[a^{(2)}] - R[a^{(3)}]R[a^{(4)}] = 0 \tag{4}
\]

for all

\( \{a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}\} \) whenever

\[
0 \leq a^{(1)} + a^{(2)} = a^{(3)} + a^{(4)} \leq S = \{S_i\}_{i=1}^n.
\]

Any \( P(x) \) with the vector \( \{S_i\}_{i=1}^n = \{2 \alpha_i\}_{i=1}^n \) of highest degrees can be written as a quadratic function of the variables \( R[a^{(i)}] \) which has the form