HAMILTONIAN SYSTEM AND THE SAINT VENANT
PROBLEM IN ELASTICITY*

Zhong Wanxie (钟万勰) Xu Xinsheng (徐新生)
Zhang Hongwu (张洪武)

(Received June 5, 1995)

Abstract
The traditional semi-inverse solution method of the Saint Venant problem, which is described in the Euclidian space under the Lagrange system formulation, is updated to be solved in the symplectic space under the conservative Hamiltonian system. It is proved in this paper that all the Saint Venant solutions can be obtained directly via the zero eigenvalue solutions and all their Jordan normal form of the corresponding Hamiltonian operator matrix.

Key words Hamiltonian system, Saint Venant problem, Symplectic, eigenvalue problem

I. Introduction

The solution of elasticity for a prismatic domain is a classical problem lasting for more than a century. Because of the complication of the partial differential equations, the traditional method of separation of variables cannot be applied. Saint Venant proposed the famous semi-inverse solution method, that some appropriate assumptions to the deformation should be made beforehand to find the solution, afterwards checking the assumptions being valid. Thereafter, the semi-inverse solution becomes the classical solution method in elasticity. The boundary conditions at the two ends of the prismatic domain can only be satisfied in the sense of static equivalent, termed the Saint Venant principle. However, the semi-inverse method is the method of try and error, and is not so ideal, that it can only find some solutions, but cannot assert if there has been no further solutions, neither how to find the remaining solutions.

Looking from the analogy theory of computational structural mechanics and optimal control, the Hamiltonian system theory can be introduced into the theory of elasticity, that the eigenfunction expansion method of the Hamiltonian operator matrix along the transverse cross-section can be developed within the symplectic geometry space, thus the solution method reaches a new level. The eigenvalue problem of the Sturm-Liouville type, which is derived from the traditional separation of variables technique within the Euclidian

* Project supported by the National Natural Science Foundation of China
1 Research Institute of Engineering Mechanics, Dalian University of Technology, Dalian 116023. P. R. China

827
space, is extended. Based on the separation of variables of the Hamiltonian system, it can be proved that among its eigenfunction-vectors there exist adjoint symplectic ortho-normalisation relationship and the corresponding expansion theorem\cite{7}. The eigenvalue zero and its corresponding eigenfunction-vectors in combination with their Jordan type eigenfunction-vectors play a specially important role\cite{9}. It will be shown in this paper that all the solutions of extensional, torsional and bending problems so far solved by the Saint Venant semi-inverse method correspond to all the solutions of the multiple eigenvalue zero solutions of the Jordan type. Thus all the semi-inverse assumptions are released, and not only the subspace spanned by all the Saint Venant solutions is given its special definite meaning, but also the exclusive argument can be shown that the eigenvalue zero corresponding extended eigensolutions have no further solution, i.e. that the Saint Venant semi-inverse method derived solutions have covered all the eigenvalue zero corresponding solutions.

II. The Fundamental Equations

The homogeneous isotropic singly connected cross-section elastic body of prismatic domain is considered. The Cartesian coordinate \((x, y, z)\) is selected that the \(z\) axis is along the longitudinal direction, and the original point locates at the central point of the cross-section \(\Omega\). \(\Omega\) is a singly connected domain, the outward normal \(\mathbf{n}\) of its boundary contour \(\partial\Omega\) has direction cosines \((l, m)\). The boundary conditions at the contour \(\partial\Omega\) are free from traction

\[
\begin{align*}
\tau_{xx} + m\tau_{yz} &= 0 \\
\sigma_x + m\tau_{xy} &= 0 \\
\tau_{yy} + m\sigma_y &= 0
\end{align*}
\]  

In this paper the notations are the same as in \[2\]. The strain displacement relations can be described as

\[
\begin{align*}
\varepsilon_z &= \frac{\partial u}{\partial x}, & \varepsilon_y &= \frac{\partial v}{\partial y}, & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{xz} &= \frac{\partial w}{\partial x}, & \gamma_{yz} &= \frac{\partial w}{\partial y} + v, & \varepsilon_z &= w
\end{align*}
\]  

where the dot represent differential with respect to \(z\), \((\cdot) = \frac{\partial}{\partial z}(\cdot)\). The \(z\) coordinate is analogous to the time coordinate. Let

\[
\varepsilon = \{\varepsilon_z, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}, \varepsilon_z\}^T \\
\sigma = \{\sigma_z, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}, \sigma_z\}^T
\]

and the stress-strain relation is given as

\[
\sigma = D\varepsilon, \quad D_8 = \begin{bmatrix}
\lambda + 2G & \lambda & 0 & 0 & 0 & \lambda \\
\lambda & \lambda + 2G & 0 & 0 & 0 & \lambda \\
0 & 0 & G & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
\lambda & \lambda & 0 & 0 & 0 & \lambda + 2G
\end{bmatrix}
\]  

(2.4)