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Symmetries and Conservation Laws of Partial Differential Equations: Basic Notions and Results

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Abstract. The main notions and results which are necessary for finding higher symmetries and conservation laws for general systems of partial differential equations are given. These constitute the starting point for the subsequent papers of this volume. Some problems are also discussed.

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0. Introduction

In this paper we present the basic notions and results from the general theory of local symmetries and conservation laws of partial differential equations. More exactly, we will focus our attention on the main conceptual points as well as on the problem of how to find all higher symmetries and conservation laws for a given system of partial differential equations. Also, some general views and perspectives will be discussed. The material of this paper is used in other papers [7]–[12] of this volume in which the general theory is applied to concrete equations of mathematical physics. This demonstrates the theory in action.

Presented here are the results on higher symmetries and conservation laws found by the author in 1975–1977 and its resumé was published in a short note [1] (see also [2]). However, the author was not successful in publishing full details until 1984. Between these years, other authors developed similar ideas (Ibragimov [5], Olver [6] etc.) in the field of symmetry theory. Tsujishita's work [17] contains some results on conservation laws which are close to ours. Higher symmetries and conservation laws for spatially one-dimensional evolution equations have been the subject of many works on the investigation of equations integrable by the inverse scattering transform method. We do not touch on these very interesting but very special topics in this paper.

In further exposition we will use the usual coordinate language for mathematical physics. However, this is not the best way to think of these substances. We omit here both motivations of basic notions (these are given in [3]) and proofs or

their indications for the presented results. An interested reader will cover this gap by consulting [4, 13, 17]. We also recommend paper [14] in which all the main technological details necessary for finding symmetries and conservation laws are demonstrated in a concrete example.

Probably the most interesting new point of the theory presented below is that in principle, it makes it possible to find all higher conservation laws for arbitrary (nonlinear) differential equations. In particular, it works effectively well in situations when the Nöther theorem, as well as other symmetry considerations, are not applicable. How it looks from the practical point of view will be clear from subsequent papers.

1. On Terminology

Below, the theory of higher local infinitesimal symmetries and conservation laws of partial differential equations is discussed. We use the adjective 'higher' to stress that the symmetries and conservation laws under consideration are described by means of expressions containing arbitrary order derivatives of quantities, entering into investigating differential equations.

The adjective 'local' is used to point out that we deal with symmetries and conservation laws which admit localizations on arbitrary domains in the space of independent variables. Foundations of nonlocal theory are considered in [15] in this volume.

The classical symmetries theory, originated by S. Lie, operates, with first-order derivatives. By speaking of 'higher symmetries', we underline the aspect which differentiates the modern theory from the classical one. Some authors use 'generalized symmetries' or 'Lie–Bäcklund transformations' in the same sense. The last term seems to be very misleading because the notion of 'Bäcklund transformation' is a concept of a quite different nature. In particular, higher symmetries are infinitesimal transformations, but Bäcklund transformations are finite ones.

Below, using the word 'symmetry', we have in mind 'higher local symmetry'.

2. Infinitely Prolonged Equations

Informally, infinitesimal symmetries are infinitesimal transformations of manifolds of infinitely prolonged equations which conserve their natural contact structures. For this reason, we consider these notions in more detail.

Let $x = (x_1, \dots, x_n)$ be independent variables and $u = (u^1, \dots, u^m)$ be dependent ones. Geometrically, this means that we deal with a smooth fibre bundle $\pi\colon E \to M$, x's are the base coordinates in it and u's are the fibre coordinates.

In some situations below, the multiindexes

$$\sigma = (i_1, \dots, i_n), \quad i = 1, 2, \dots, n,$$