On the variational boundary integral equations in elastodynamics with the use of conjugate functions

H.D. BUI
Electricité de France, Centre de Clamart & Ecole Polytechnique,
Laboratoire de Mécanique des Solides, 91128 Palaiseau, France

Received 26 September 1990

Key words: variational B.I.E., conjugate functions, a posteriori error, symmetry, regular kernels

Abstract. This paper proposes a Variational Boundary Integral Equation for time harmonic elasticity, using conjugate functions. A bilinear hermitian form for the variational formulation, as well as an a posteriori error indicator are proposed. The method does not involve hypersingular integrals in the finite part sense and preserves the symmetrical structures of equations.

1. Introduction

In a recent paper [1], we introduced a formulation of the boundary integral equation method for elastostatics with the aid of conjugate functions. The conjugate functions are the Helmholtz potentials \( \phi(x) \), \( H(x) \) and the Beltrami tensor \( B(x) \) which allow the representation of the displacement and the stress respectively by

\[
\begin{align*}
\mathbf{u} &= \nabla \phi + \text{rot} \mathbf{H}, \\
\mathbf{\sigma} &= R^*R \mathbf{B},
\end{align*}
\]

where \( R \) is the right-curl operator, and \( R^* \) its adjoint, i.e. the left-curl operator, as will be further defined. The strain \( \mathbf{\varepsilon} = D^*\mathbf{u} \equiv (1/2)(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \mathbf{u} \) and the stress \( \mathbf{\sigma} \) are linked by the elasticity law \( \mathbf{\sigma} = \mathbf{A} \mathbf{\varepsilon} \) which will be assumed linear and isotropic (\( \mathbf{A} \): transposition). In classical methods of solving boundary value problems of elasticity, e.g. the primal problem, one starts from the displacement \( \mathbf{u} \) (or the potentials \( \phi, H \)) and uses the mappings

\[
(\phi, H) \rightarrow \mathbf{u} \rightarrow \mathbf{\varepsilon} \rightarrow \mathbf{\sigma} \rightarrow \mathbf{f} \equiv D\mathbf{\sigma} \equiv -\text{div} \mathbf{\sigma} = 0 \text{ (no body force)}.
\]

In the dual formulation, one may use either the stress \( \mathbf{\sigma} \) (with the constraint \( \mathbf{D}\mathbf{\sigma} = 0 \)) or the Beltrami tensor \( B \) (without internal constraint) and consider the mappings \( B \rightarrow \mathbf{\sigma} \rightarrow \mathbf{\varepsilon} \rightarrow \mathbf{\eta} = 0 \text{ (no incompatibility)} \), Fig. 1.
The diagram of Fig. 1, given in Tonti [2], reveals the symmetrical structure of quasistatic elasticity. There are constraint relations between the boundary values of conjugate functions and derivatives due to boundary conditions which are not represented in the diagram. Classical boundary integral equations (B.I.E.) are generally based on the primal formulation. In the absence of body force \((f = 0)\), the B.I.E. deriving from the Betti's reciprocal theorem and the Green function constitute the so-called direct method for determining the boundary unknowns \(u\) and \(T(u) \equiv \sigma(u) \cdot n\) (traction vector). A direct discretization of the singular integral equations, by the collocation method, yields a non-symmetric linear system of equations. To restore the symmetry, some authors [3, 4, 5] proposed a variational formulation of the direct B.I.E. in which the boundary values of \(u\) and \(T\) are considered as independent variables. Such a mixed formulation of the Reissner–Hellinger type for dual variables \((u, T)\) is based on the system of two equations: the B.I.E. from the displacement formulation with Cauchy principal value kernels and the B.I.E. from the stress formulation with hypersingular kernels in the Hadamard finite part sense. The variational formulation of the indirect B.I.E. using Kupradze's potentials [9] has been proposed by Nedelec [10]. His formulation is symmetric and non-hypersingular. The formulation by conjugate functions \(X = (\phi, H, B)\) given in [1] is also symmetric and non-hypersingular. Moreover, it provided an *a posteriori* error estimate of numerical solutions. Following an idea originated by Ladevèze [6, 7] for the finite elements method, we considered in [1] that the solution of the boundary value problem can be obtained by solving the problem: Minimize \(A(X, X)\) in the set of admissible fields \(X\), where \(A(X, X)\) is the residual of the constitutive law with the energy norm,

\[
A = \int_{\Omega} (\sigma - \Lambda \varepsilon) : (\Lambda^{-1} \sigma - \varepsilon) \, d\omega.
\]  

(3)

The set of statically admissible fields \(\sigma\) and kinematically admissible fields \(\varepsilon\) must satisfy some additional boundary constraints determined by the boundary values of the problem under consideration. By its definition, the