ABSTRACT. Building on previous work of A. Camacho, we give necessary and sufficient conditions for the existence of a cardinal utility function to represent, through summation, a preference relation on sequences of alternatives.

1. INTRODUCTION

Till recently, there were mainly three ways to derive cardinal utility. One is the approach, using strength of preference as a primitive. A second approach uses lotteries. Thirdly there is the approach where alternatives have several coordinates, and the utility function is a sum of coordinate functions.

Recently Camacho came with a new approach, the repetitions approach. For a careful exposition of this approach, a comparison to other approaches, and an explanation of its intuitive virtues, the reader is referred to Camacho [1-4]. The purpose of this paper is to use the ideas of Camacho to give a set of necessary and sufficient conditions, alternative to his set, and to give some supplement to his work. Where Camacho works with finite sequences, we use infinite sequences with tails $\alpha_0$ ("zero"); in Section 3 we shall show that our set-up is in fact equivalent to Camacho's. We only work with these infinite sequences for their convenience in our present mathematical work.

We assume we have a nonempty set $\mathcal{A}$ of alternatives, with one special element $\alpha_0$, the "receive nothing" alternative. By $\mathcal{X} \subset \mathcal{A}^{\mathbb{N}}$ we denote the set of those infinite sequences $x = (x_j)_{j \in \mathbb{N}}$, for which

$$N_x := \sup \{\{0\} \cup \{j \in \mathbb{N} : x_j \neq \alpha_0\}\}$$

is finite, so $x$ has a "tail", constant $\alpha_0$. Furthermore we assume a binary relation $\succeq$ on $\mathcal{X}$, called preference relation, present. Usual notations are $x \preceq y$ for $y \succeq x$, $x \succeq y$ for $x \succeq y$ & not $y \succeq x$, $x \prec y$ for $y \succ x$, and
\( x \preceq y \) for \( x \succeq y \) \& \( y \succeq x \). \( \succeq \) is a \textit{weak order} if it is transitive and complete (\( x \succeq y \) or \( y \succeq x \), for all \( x, y \in X \)).

Our purpose is to find a function
\[
u : \mathcal{A} \to \mathbb{R} \text{ s.t. } x \succeq y \iff \sum_{j=1}^{\infty} [u(x_j) - u(y_j)] \geq 0.
\]

For such a function to exist, \( \succeq \) must certainly satisfy the following four axioms, as can be checked straightforwardly and is not elaborated here.

**AXIOM 1.** \( \succeq \) is a weak order.

**AXIOM 2 (The Permutation Axiom).** For all \( x, y \in \mathcal{X}, N \in \mathbb{N}, \) permutations \( \pi \) on \( \{1, \ldots, N\} \), s.t. \( x_j = y_{\pi(j)} \) for all \( j \leq N \), \( x_j = y_j \) for all \( j > N \); we have \( x \succeq y \).

(A reordering of alternatives does not change desirability).

**AXIOM 3 (The Independence Axiom).** For all \( x, y, x', y' \in \mathcal{X}, i \in \mathbb{N}, \) s.t. \( x_i = y_i, \ x'_i = y'_i, \ x_j = x'_j \) and \( y_j = y'_j \) for all \( j \neq i \), we have \( x \succeq y \iff x' \succeq y' \).

(The preference between \( x \) and \( y \) is independent of coordinates \( i \) at which \( x \) and \( y \) are equal.)

**AXIOM 4 (The Archimedean Axiom).** For all \( x, y, v, w \in \mathcal{X} \) with \( x \succ y, v \succ w \), there exists \( M \in \mathbb{N} \) s.t. \( p \succ q \) where \( p_{kN_x + j} = x_j \) for all \( 0 \leq k \leq M - 1, 1 \leq j \leq N_x \), \( p_{MN_x + j} = w_j \) for all \( 1 \leq j \leq N_w \), and \( p_n = x^0 \) for all \( n > MN_x + N_w \); and where \( q_{kn_y + i} = y_i \) for all \( 0 \leq l \leq M - 1, 1 \leq i \leq N_y \), \( q_{MN_y + i} = v_i \) for all \( 1 \leq i \leq N_v \), and \( q_m = x^0 \) for all \( m > MN_y + N_v \).

(The difference between \( v \) and \( w \) can be compensated by a sufficient number of differences between \( x \) and \( y \).)

Constructions such as that of \( p \) above will more often be carried out in the sequel. One can imagine the "untailed" part of \( p \) to consist of \( M \) replicas of the "untailed" part of \( x \), followed by one replica of the "untailed" part of \( w \). Axiom 4 has not been used by Camacho, but he indicated it more or less in Section 2.1, page 364, (d), in [3].