ABSTRACT. Building on previous work of A. Camacho, we give necessary and sufficient conditions for the existence of a cardinal utility function to represent, through summation, a preference relation on sequences of alternatives.

1. INTRODUCTION

Till recently, there were mainly three ways to derive cardinal utility. One is the approach, using strength of preference as a primitive. A second approach uses lotteries. Thirdly there is the approach where alternatives have several coordinates, and the utility function is a sum of coordinate functions.

Recently Camacho came with a new approach, the repetitions approach. For a careful exposition of this approach, a comparison to other approaches, and an explanation of its intuitive virtues, the reader is referred to Camacho [1–4]. The purpose of this paper is to use the ideas of Camacho to give a set of necessary and sufficient conditions, alternative to his set, and to give some supplement to his work. Where Camacho works with finite sequences, we use infinite sequences with tails $x_0$ ("zero"); in Section 3 we shall show that our set-up is in fact equivalent to Camacho's. We only work with these infinite sequences for their convenience in our present mathematical work.

We assume we have a nonempty set $\mathcal{A}$ of alternatives, with one special element $x^0$, the "receive nothing" alternative. By $\mathcal{X} \subset \mathcal{A}^\infty$ we denote the set of those infinite sequences $x = (x_j)_{j \in \mathbb{N}}$, for which

$$N_x := \sup \{ \{0\} \cup \{ j \in \mathbb{N} : x_j \neq x^0 \}\}$$

is finite, so $x$ has a "tail", constant $x^0$. Furthermore we assume a binary relation $\succeq$ on $\mathcal{X}$, called preference relation, present. Usual notations are $x \preceq y$ for $y \succeq x$, $x \succeq y$ for $x \succeq y$ & not $y \succeq x$, $x < y$ for $y > x$, and
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$x \approx y$ for $x \succeq y$ & $y \succeq x$. $\succeq$ is a weak order if it is transitive and complete ($x \succeq y$ or $y \succeq x$, for all $x, y \in X$).

Our purpose is to find a function

$$u: \mathcal{A} \to \mathbb{R} \text{ s.t. } x \succeq y \iff \sum_{j=1}^{\infty} [u(x_j) - u(y_j)] \geq 0.$$  

For such a function to exist, $\succeq$ must certainly satisfy the following four axioms, as can be checked straightforwardly and is not elaborated here.

**AXIOM 1.** $\succeq$ is a weak order.

**AXIOM 2 (The Permutation Axiom).** For all $x, y \in \mathcal{X}, N \in \mathbb{N}$, permutations $\pi$ on $\{1, \ldots, N\}$, s.t. $x_j = y_{\pi(j)}$ for all $j \leq N$, $x_j = y_j$ for all $j > N$; we have $x \approx y$.

(A reordering of alternatives does not change desirability).

**AXIOM 3 (The Independence Axiom).** For all $x, y, x', y' \in \mathcal{X}, i \in \mathbb{N}$, s.t. $x_i = y_i$, $x'_i = y'_i$, $x_j = x'_j$ and $y_j = y'_j$ for all $j \neq i$, we have $x \succeq y \iff x' \succeq y'$.

(The preference between $x$ and $y$ is independent of coordinates $i$ at which $x$ and $y$ are equal.)

**AXIOM 4 (The Archimedean Axiom).** For all $x, y, v, w \in \mathcal{X}$ with $x \succeq y$, $v \succeq w$, there exists $M \in \mathbb{N}$ s.t. $p \succeq q$ where $p_{kN_x + j} = x_j$ for all $0 \leq k \leq M - 1$, $1 \leq j \leq N_x$, $p_{MN_x + j} = w_j$ for all $1 \leq j \leq N_w$, and $p_n = x^0$ for all $n > MN_x + N_w$; and where $q_{kN_y + i} = y_i$ for all $0 \leq l \leq M - 1$, $1 \leq i \leq N_y$, $q_{MN_y + i} = v_i$ for all $1 \leq i \leq N_v$, and $q_m = x^0$ for all $m > MN_y + N_v$.

(The difference between $v$ and $w$ can be compensated by a sufficient number of differences between $x$ and $y$.)

Constructions such as that of $p$ above will more often be carried out in the sequel. One can imagine the "untailed" part of $p$ to consist of $M$ replicas of the "untailed" part of $x$, followed by one replica of the "untailed" part of $w$. Axiom 4 has not been used by Camacho, but he indicated it more or less in Section 2.1, page 364, (d), in [3].