Global Solutions of the Navier-Stokes Equation with Strong Viscosity

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Abstract: Following Ebin and Marsden the Navier-Stokes equation is viewed as a perturbation of a geodesic flow on the group of volume preserving diffeomorphisms on a compact Riemannian manifold. Existence and uniqueness of bounded solutions for all positive time is shown by taking a higher order diffusion term.

Key words: Navier-Stokes equation, diffeomorphism groups, geodesic flow

MSC 1991: 58D30, 35Q30

1. Introduction

In the paper [4] Ebin and Marsden study the flow of a viscosity free fluid via a geometrical approach of Arnold [1]. The approach is to formulate the equation for the fluid flow (i.e. the Euler equation) as a geodesic dynamical system on the infinite dimensional manifold of volume preserving diffeomorphisms of the vessel in which the fluid flows. By studying the regularity of the geometrical structure on this manifold, they deduce that the Euler equation does have a solution, which lives for a certain time. They also treat the flow of a viscous fluid (whose equation is the Navier-Stokes equation), by regarding the viscosity (diffusion) term as a perturbation of the Euler equation.

This present paper is based on [4]; our aim is to show that the phenomenon of viscosity can prevent the solution from "exploding" so that it continues for all time. That viscosity should do this is reasonable because diffusions are generally contractive; the solution to the equation with only the diffusion term (i.e. the Stokes equation, whose solution is "slow viscous flow"), tends to be more pleasant than its initial condition. Unfortunately, to make our analysis work, we have to take a stronger viscosity term than is usual in the Navier-Stokes equation; we must raise the usual viscosity term to the power $r$, where $r > m/4 + 1/2$, and $m$ is the dimension of the space in which the fluid flows.

The following is a statement of our main result, expressed in the geometrical language of [4]; we will briefly review the geometrical background of our work in Section 2.

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Theorem 1. For \( s > m/2 + 1, \ r > s/2, \ v > 0, \ q > 0, \) and any initial condition \( \xi_0 \in TV \), the following modified form of the Navier-Stokes equation has a unique solution, which is defined for all positive time and is bounded:

\[
\dot{\xi}_t = Z(\xi_t) - [vA(\xi_t) + qA(r(\xi_t))]'.
\]

(1)

We prove this theorem in Section 3. Our notation is the following:

\( M \) is a smooth compact oriented riemannian manifold of dimension \( m \) without boundary. This is the space ("vessel") in which the fluid flows. In [4] the treatment of the Euler equation is adopted to deal with the case when \( M \) has a boundary, but the work on the Navier-Stokes equation would seem to be more difficult to adopt. For this it might be appropriate to introduce a "vorticity creating operator" into the fluid flow. See [2], [9].

\( \mu \) is the riemannian volume on \( M \).

\( D^s_\mu \) is the manifold of Sobolev \( H^s \) smooth diffeomorphisms of \( M \) preserving volume and orientation.

\( Z : TTV \rightarrow TV \) is the "weak" spray on \( D^s_\mu \), i.e. the vector field whose flow is the "weak" geodesic flow. The equation \( \dot{\xi}_t = Z(\xi_t) \) corresponds to the Euler equation.

\( \Delta \) is the Laplacian on divergence free vector fields over \( M \). The equation \( \dot{\xi}_t = -v[\Delta(\xi_t)]' \) corresponds to the Stokes equation.

\([\ldots]'\) denotes the vertical lift from \( TV \) to \( TV \) via the "weak" connection on \( D^s_\mu \) (see Section 2).

Notes.

(i) Setting \( q = 0 \) in (1) yields the usual form of the Navier-Stokes equation, but as we have said, we require \( q > 0 \) in our work. However, for given large \( r \), and for sufficiently small \( q \), the term \( q\Delta r \) in (1) is significant only for high order eigenvectors of \( \Delta \). Thus, to decide whether \( q = 0 \) or merely \( q \) small is appropriate in the Navier-Stokes equation, one would have to consider the fluid at a molecular level.

(ii) Classical techniques [8] have shown that for \( m = 2 \), the usual form of the Navier-Stokes equation (i.e. with \( r = 1 \)) does have a solution defined for all positive time. We could obtain this if we could relax our hypothesis to "\( s > m/2 + 1, r \geq s/2 \)."

2. The Geometry of \( D^s_\mu \)

In this section we summarize the results concerning the geometrical structures on \( D^s_\mu \) on which our work is based. Most of these results are proved in [3] or [4].

If \( M \) is as above, then for \( s > m/2 + 1 \), the collection \( D^s \) of Sobolev \( H^s \) smooth diffeomorphisms of \( M \) (i.e. which are \( H^s \) smooth when expressed in charts) is a smooth manifold, modeled on the hilbertible space \( \Gamma^s_{id}(M, TM) \) of \( H^s \) vector fields on \( M \). In a neighbourhood of \( id \in D^s \) we have a canonical chart map

\[ \text{Exp}_{id} : \Gamma^s_{id}(M, TM) \rightarrow D^s : X \rightarrow \exp X \]

where \( \exp : M \rightarrow TM \) is the exponential map of the riemannian structure on \( M \). In fact, the tangent space \( T_{id}D^s \) can be identified naturally with \( \Gamma^s_{id}(M, TM) \) by identifying the vector \( \frac{d}{dt}|_{t=0} \phi_t \in T_{id}D^s \) with the map \( M \rightarrow TM : x \rightarrow \frac{d}{dt}|_{t=0} \phi_t(x) \), where \( \phi_t \) is a smooth curve in \( D^s \) with \( \phi_0 = id \). In a similar way, \( T_{\phi}D^s \) can be identified with \( \Gamma^s_{\phi}(M, TM) = \{ H^s \text{ maps } X : M \rightarrow TM \text{ such that } \pi \circ X = \phi \} \), and for