A Globally Convergent Method for Semi-Infinite Linear Programming

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Abstract. This paper presents a globally convergent method for solving a general semi-infinite linear programming problem. Some important features of this method include: It can solve a semi-infinite linear program having an unbounded feasible region. It requires an inexact solution to a nonlinear subproblem at each iteration. It allows unbounded index sets and nondifferentiable constraints. The amount of work at each iteration k does not increase with k.

Key words: Semi-infinite linear programming, optimality, convergence.

1. Introduction

The primal problem of semi-infinite linear programming is defined as

(SIP) maximize $c^T x$
subject to $a(u)^T x - b(u) \leq 0$ for all $u \in U$,

where $c, x \in \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ is an index set containing infinitely many points, $a : U \rightarrow \mathbb{R}^n$, and $b : U \rightarrow \mathbb{R}^1$. Without loss of generality, we assume that $c$ is a unit vector, $a(u) \neq 0$ for all $u \in U$, and $\sup\{\|(a(u), b(u))\|_\infty : u \in U\} < \infty$.

There are many practical as well as theoretical problems in which the constraints depend on time or space and thus can be formulated as semi-infinite programs. The question of how to compute numerically a solution of a semi-infinite program has received increasing attention (see, e.g., Ferris and Philpott (1989), Hu (1990), Kortanek and No (1993), and Todd (1994)). For a recent extensive survey on semi-infinite programming theory, methods, and applications, one may consult Hettich and Kortanek (1993). Some common restrictions on (SIP) imposed by most existing methods are that the feasible region must be bounded and the index set $U$ must be compact and have a nice structure. Many methods also require, at each iteration $k$, finding an exact solution of the nonlinear program $\sup\{a(u)^T x^k - b(u) : u \in U\}$. In this paper we present a globally convergent method for solving a general semi-infinite linear programming problem. Some important features of this method include: It can solve a semi-infinite linear program having an unbounded feasible region. It requires an inexact solution to a nonlinear subproblem at each iteration. It allows unbounded index sets (e.g., $U = \{1, 2, \ldots \}$) and nondifferentiable constraints. The amount of work at each iteration $k$ does not increase with $k$. 
2. Preliminaries

Let $S = \{x : a(u)^T x - b(u) \leq 0 \text{ for all } u \in U\}$ denote the feasible region of (SIP), $v^* = \sup \{c^T x : x \in S\}$ denote the optimal value of (SIP), and $S^* = \{x \in S : c^T x = v^*\}$ denote the set of optimal solutions of (SIP). We assume that $S$ is nonempty, which can be verified by solving a phase (I) problem (Gustafson 1983). Note that $S^*$ may be empty if $S$ is unbounded.

Let $\| \cdot \|$ be the Euclidean norm and $d(x) = \min\{ \|x - y\| : y \in S\}$ be the Euclidean distance from $x$ to $S$.

For any real number $t$, we define

$$t^+ = \begin{cases} t, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Let $r(x) = \sup\{a(u)^T x - b(u)^+ : u \in U\}$ be the "biggest violation" by $x$. It is easy to verify that (i) $0 \leq r(x) < \infty$ for all $x \in \mathbb{R}^n$, (ii) $r(x) = 0$ if and only if $x \in S$, (iii) $r(x) \geq \sup\{a(u)^T x - b(u) : u \in U\}$ for all $x \in \mathbb{R}^n$, (iv) $r(x) = \sup\{a(u)^T x - b(u) : u \in U\}$ if $x \notin S$, and (v) $r(x)$ is a continuous convex function on $\mathbb{R}^n$.

$r(x)$ measures how much $x$ violates the constraints and $d(x)$ measures how far $x$ is from the feasible region. If the feasible region $S$ is ill-conditioned, then it is possible to find a sequence $\{x_k : k = 1, 2, \ldots\}$ such that $\lim_{k \to \infty} r(x_k) = 0$ and $\lim_{k \to \infty} d(x_k) = \infty$ (Hu and Wang 1989). In this situation, the task of computing numerically a solution to (SIP) becomes very difficult. Hence, in the rest of our discussion, we assume that $S$ satisfies the following condition:

**CONDITION A.** There exists a constant $\tau > 1$ such that $d(x) \leq \tau r(x)$ for all $x \in \mathbb{R}^n$.

The existence and computation of $\tau$ are discussed in Hu and Wang (1989). For example, if there exists a unit vector $\bar{x}$ and a positive number $\beta$ such that $a(u)^T \bar{x} \geq \beta$ for all $u \in U$, then $\tau = \beta^{-1}$. Note that $S$ is unbounded in this case. For a second example, if $S$ is bounded by $M$ and $b(u) > \beta > 0$ for all $u \in U$, then $\tau = \delta^{-1} M$. The existence of a relatively small $\tau$ ensures that if $x$ almost satisfies the constraints, then $x$ is close to (in Euclidean distance) the feasible region.

The basic idea of our method is: Let $\epsilon_k > 0$, $\lambda_k > 0$, and $x^k$ be the current iterate. Find a constraint whose index $u_k$ is an $\epsilon_k$ solution of the nonlinear program

$$\sup\{a(u)^T (x^k + \lambda_k c) - b(u) : u \in U\}.$$ 

If $x^k + \lambda_k c$ satisfies this constraint, then $x^k + \lambda_k c$ is close enough to the feasible region and the method can focus on improving the objective function value by letting $x^{k+1} = x^k + \lambda_k c$. Otherwise, the method takes care of optimality and feasibility by letting $x^{k+1}$ be the projection of $x^k + \lambda_k c$ on $a(u_k)^T x = b(u_k)$. The sequence $\{x^k : k = 1, 2, \ldots\}$ will converge to an optimal solution of (SIP) if an optimal solution exists and the sequences