APPLICATIONS OF PSEUDO-BOOLEAN METHODS TO ECONOMIC PROBLEMS*

ABSTRACT. The use of Boolean algebra in logic, switching and automata theory, coding and other technically oriented areas is well known. The role of this paper is to show that Boolean algebra can be instrumental in taking economic decisions.

By a pseudo-Boolean function, a real-valued function with bivalent (0–1) variables will be understood. The symbols 0 and 1 will stay both for their logical meaning and their arithmetical value.

The basic problems which arise frequently in connection with pseudo-Boolean functions are: (1) solution of systems of equations and/or inequalities involving only pseudo-Boolean functions, (2) problems of determining the maximum or the minimum of a free pseudo-Boolean function, or of a pseudo-Boolean function whose variables are subject to constraints; (3) problems of finding the minimax or the maximin of a pseudo-Boolean function.

The basic problems outlined above are exemplified on the case of a company wishing to locate a number of service stations, which – under different assumptions – lead to the above formulated models.

Pseudo-Boolean programming is a collection of techniques enabling the analysis of situations where decisions of the type 'yes' or 'no' are to be taken, and effectively determining the optimal decisions of this category.

Most of the problems which were studied until now by pseudo-Boolean programming belonged to mathematics themselves, and mostly to graph theory. There are also a number of other problems which were studied by this approach, as for instance problems related to switching circuits, coding theory and operations research.

Although it is apparent that numerous economic problems can be formulated as problems of pseudo-Boolean programming, until now only very few of them were studied under this aspect.

The role of this paper is to point out the usefulness of pseudo-Boolean programming for the solution of economic problems, where 'yes' or 'no' decisions are to be taken.

The techniques which are used for the solution of pseudo-Boolean programming problems will not be described here; however, the major types of such problems will be outlined. For the details of pseudo-Boolean programming, the reader is referred to Hammer and Rudeanu (1968).

Theory and Decision 1 (1971) 296–308. All Rights Reserved
Copyright © 1971 by D. Reidel Publishing Company, Dordrecht-Holland
To any 'yes' or 'no' decision $D$ we can associate a bivalent variable $d$, by putting
\[
d = \begin{cases} 
1 & \text{if the decision } D \text{ is 'yes'} \\
0 & \text{if the decision } D \text{ is 'no'}.
\end{cases}
\]

If we have to take a single decision, or if we have to take several independent decisions, the analysis of its (or their) outcome(s) is simple. We just have to consider the two possible alternatives, and choose the best one.

The situation is much more complex, if we have to take several decisions, which are not independent. In order to know the outcomes in every possible alternative, we have to consider $2^n$ possibilities (where $n$ is the number of 'yes'-'no' decisions we have to take), and choose the best one of them. Obviously for a great $n$ this becomes impossible.

Let us associate now to the different outcomes of such a set of decisions $d_1, \ldots, d_n$, a set $w_1, \ldots, w_m$ of real numbers. The significance of the $w$'s can vary, they may stand for costs, profits, durations (time), consumptions, values, etc. As every system of bivalent values $(d_1, \ldots, d_n)$ determines a system of real values $(w_1, \ldots, w_m)$, we may say that the $w$'s are functions of the $d$'s. Such functions will be called pseudo-Boolean functions.

Hence, by a pseudo-Boolean function
\[
f(x_1, \ldots, x_n)
\]
we shall mean a real-valued function with bivalent (0, 1) variables.

One of the specific features of a pseudo-Boolean function, is that it can be always written as a polynomial. Moreover, due to the fact that $0.0=0$ and $1.1=1$, every such polynomial, will be linear in each of its variables.

For example, the pseudo-Boolean function defined by the following tableau:

\[
\begin{array}{c|c|c}
  x_1 & x_2 & f(x_1, x_2) \\
  \hline
  0 & 0 & 2 \\
  0 & 1 & -3 \\
  1 & 0 & 7 \\
  1 & 1 & -1 \\
\end{array}
\]

has the following polynomial expression:
\[
f(x_1, x_2) = 2 + 5x_1 - 5x_2 - 3x_1x_2.
\]