Routes and paths of comparison and choice

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Abstract

This paper studies aspects of sequential choice (and elimination) through the ‘route-wise’ application of choice. (A choice is a mapping of the subsets of a set X into their respective subsets.) This approach sheds some further light on the ‘path-independence’ of choice, as well as on the logical structure of several rationality criteria for choice, as expressed through the properties of the comparison (or preference relation) revealed by a choice. The results bear particular relevance to the theory of collective choice.

When K.J. Arrow argued for what he called ‘collective rationality’ he emphasized that this would include the ‘independence of the final choice from the path to it’ (1963: 120). This does not really specify the nature of the desired independence too clearly, as was also observed by Charles Plott (1973: 1075).

One possible notion of ‘path-independence’ of choice has been formalized by Plott (1973) and used by himself and others (see, e.g., D.H. Blair, 1975; R.P. Parks, 1976; J. Richelson, 1977; A.K. Sen, 1977; M.R. Sertel, 1976; Sertel and Van der Bellen, 1979b; and Van der Bellen, 1976). In two previous papers (Sertel and Van der Bellen, 1976, 1979a) we presented seven characterizations of this notion of path-independence of choice, and two more characterizations are given by Sertel (1978). In fact, sixteen characterizations of path-independence are logically classified in Sertel and Van der Bellen (1979b), one of them being due to Parks (1976) and another to Blair et al. (1976). Here, another approach to the analysis of choosing in ‘stepwise’, ‘sequential’, or ‘piecemeal’ fashion is pursued, using the notion of what we call the ‘routewise application’ of choice. The ensuing theory, although developed in its own right, incidentally provides further insight

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into the conditions of 'binariness' and 'consistency' for a choice, as well as offering yet another characterization of path-independence.

We proceed as follows: First, in section I, we introduce basic notions relating to choices and their routewise applications. Then, in section II, we develop the theory. Finally, in section III, we discuss our present approach, de-emphasizing 'rationality' (often implying, and in some authors' terminology equivalent to, binariness) of choice.

I. Preliminaries

Throughout, \( X \) is a non-empty set, \( \leq \) is a well-ordering of \( X \), and \( [A] \) denotes the power set of any \( A \subseteq X \). \( F[X] \) stands for the set of finite subsets of \( X \). \( E \) is any family of sets satisfying

\[
F[X] \subseteq E \subseteq [X] \tag{1}
\]

\[
A \in E \Rightarrow [A] \subseteq E \tag{2}
\]

\[
A, B \in E \Rightarrow A \cup B \in E \tag{3}
\]

i.e., as a family of subsets of \( X \), \( E \) is closed under finite union and finite intersection, owns all the finite subsets of \( X \) and contains the power set of any of its elements.

1. Examples

1.1 Let \( E = [X] \), as in Sertel and Van der Bellen (1976, 1979a, 1979b).

1.2 Let \( E = F[X] \).

1.3 Take \( X \) finite (as in Plott, 1973), so that \( F[X] = E = [X] \).

1.4 Whenever \( X \) is a topological space, let \( E = K[X] \), where \( K[X] \) stands for set of relatively compact subsets of \( X \).

We extend the notion of a choice or shrinking (e.g., as in Plott, 1973; or Sertel and Van der Bellen, 1980): a choice (on \( E \)) is any transformation \( s: E \rightarrow E \) of \( E \) such that

\[
A \in E \Rightarrow sA \subseteq A \tag{4}
\]

We denote the set of all choices \( s: E \rightarrow E \) by \( \Sigma \).

2. Examples

2.1 Let \( s = 1 \), where \( 1 \) stands for the identity transformation of \( E \): \( 1A = A \) (\( A \in E \)).

2.2 Let \( s': E \rightarrow U \{A | A \in E \} \) be a choice function in the sense of the