A NEW CHARACTERIZATION OF
CONVEX PLATES OF CONSTANT WIDTH

ABSTRACT. A convex plate $D \subset \mathbb{R}^2$ of diameter 1 is of constant width 1 if and only if any two perpendicular intersecting chords have total length $\geq 1$.

1.

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a convex body of diameter 1. We say that $D$ has the property (P) if any $n$ mutually perpendicular chords, having a common point, have total length $\geq 1$. For $n = 2$, this property has been introduced by Schmitz [4]. He has shown that a circle as well as a Reuleaux triangle of diameter 1 have this property (even with strict inequality for non-degenerate chords), and further he stated that only plates of constant width can have property (P). We shall continue these considerations with the following statements.

THEOREM 1. A convex plate $D \subset \mathbb{R}^2$ of diameter 1 has property (P) if and only if it is of constant width 1. Moreover, if $D$ has constant width 1, in property (P) we have strict inequality for non-degenerate chords.

PROPOSITION. Let a convex body $D \subset \mathbb{R}^n$, $n \geq 2$, of diameter 1 satisfy the property (P) (for non-degenerate chords). Then $D$ is of constant width 1.

There remains the open question of whether for $n \geq 3$ we have an analogous equivalence. (One sees easily that the positive answer for $\mathbb{R}^{n+1}$ implies the same for $\mathbb{R}^n$. Namely, $D \subset \mathbb{R}^n$ can be induced in a convex body $D' \subset \mathbb{R}^{n+1}$ of constant width 1 (cf. Chakerian and Groemer [1]), and by Lemma 1 we may consider chords of $D'$ with a common endpoint, one parallel to the $(n + 1)$st basis vector and of length 0.) Anyway, a ball of diameter 1 in $\mathbb{R}^n$ satisfies property (P). To see this, we may clearly restrict ourselves to the case when the common point of the chords lies on the boundary, in which case

$$\sum_{i=1}^{n} |P_iQ_i| = \sqrt{\sum_{i=1}^{n} |P_iQ_i|^2} = 1.$$
2.

We shall start by proving the proposition along the lines of the hint of Schmitz [4].

Proof. We denote by $H(u)$ the supporting plane of $D$ with outer normal $u$. Let us suppose for some $u$ that $\dim(D \cap H(u)) < n - 1$, and $P \in D \cap H(u)$. Let $PQ(u)$ be the chord of $D$ through $P$ with direction $u$. Suppose $|PQ(u)| < 1$, and denote by $H_\epsilon(u)$ the translate of $H(u)$ by a distance $\epsilon$, translated inwards $D$. Then the width of $D \cap H_\epsilon(u)$ tends to 0, if $\epsilon \to 0$. Hence, through $H_\epsilon(u) \cap PQ(u)$ we can choose $n - 1$ mutually perpendicular chords of $D$, lying in $H_\epsilon(u)$, whose total length is arbitrarily small provided $\epsilon$ is sufficiently small. These $n - 1$ chords, together with $PQ(u)$, show that property $(P)$ does not hold. By this contradiction, for each $u$ with $\dim(D \cap H(u)) < n - 1$ and each $P \in D \cap H(u)$ we have $|PQ(u)| = 1$. This implies that the width of $D$ in direction $u$ is 1.

Let us now consider the direction set $\{u \mid \dim(D \cap H(u)) = n - 1\}$. Since these directions correspond to disjoint non-empty open sets of the boundary of $D$, they constitute an at most countable set. In particular, this set contains no non-empty open subset of the unit sphere $S^{n-1}$. For the width function $w(u)$ we have $w(u) = 1$ on the complement of this set. Hence, by continuity $w(u) = 1$ on all $S^{n-1}$, i.e. $D$ is of constant width 1. $\square$

We see that here the property $(P)$ was used only in the limit, $n - 1$ chords tending to 0 (or equal to 0).

LEMMA 1. Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a convex body, and let $P_1Q_1, \ldots, P_nQ_n$ be mutually perpendicular chords of $D$, having a common point $O$. (Degeneration into points of tangency with lines of direction $P_iQ_i$ is admitted, but in this case we consider the direction of $P_iQ_i$ as fixed, and perpendicularity is meant in this manner.) Then $\Sigma |P_iQ_i|$ attains its minimum in a case where $O \in \text{Bd} \ D$ and $O$ is an endpoint of each chord $P_iQ_i$, and if $D$ is strictly convex, only in such a case.

Proof. Let us fix the chord $P_nQ_n$ and translate all other chords $P_iQ_i$ parallel to themselves, the common point $O$ of all these chords traversing the whole segment $P_nQ_n$. Then each length $|P_iQ_i|$ $(1 \leq i \leq n - 1)$ is a concave function of the position of $O$ on $P_nQ_n$, and is even strictly concave for $D$ strictly convex. Hence the same holds for $\Sigma_{i=1}^{n-1} |P_iQ_i|$, and thus the minimum is attained for $O = P_n$ or $O = Q_n$, and for strictly convex $D$ this minimum cannot be attained elsewhere. Clearly, for strictly convex $D$ the point $O$ must be an endpoint of each other chord $P_iQ_i$, as well. For general $D$ observe that if $O$ was originally an endpoint of some chord $P_iQ_i$, $1 \leq i \leq n - 1$, it remained so by moving $O$ along $P_nQ_n$ provided $O$ was a relatively interior point of $P_nQ_n$. 