ON THE IDENTIFICATION OF NATURAL MODULES FOR SYMPLECTIC AND LINEAR GROUPS DEFINED OVER ARBITRARY FIELDS

Dedicated to Professor J. Tits on the occasion of his sixtieth birthday

ABSTRACT. Let $G$ be $\text{SL}_n(k)$ or $\text{Sp}(2n,k)$, $k$ a field. Then a criterion for a $\mathbb{Z}G$-module $V$ to be the direct sum of natural, resp. natural and dual, $kG$-modules is given. In fact this criterion holds for perfect central extensions of $G$ generated by 'k-root subgroups'. This has an application to the classification of non-simple groups generated by 'k-root subgroups'.

1. INTRODUCTION

Let $k$ be a field and $\Sigma$ a class of abelian subgroups generating the group $G$. Then $\Sigma$ is a class of $k$-root subgroups of $G$, if the following holds:

(1) For $A, B \in \Sigma$ one of the following holds:
   (a) $[A, B] = 1$.
   (b) $\langle A, B \rangle \cong (P)\text{SL}_2(k)$ and $A, B$ are full unipotent subgroups of $\langle A, B \rangle$ (i.e. a conjugate of $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in k \right\}$).
   (c) $\langle A, B \rangle$ is special with $[a, B] = [b, A] = [A, B] \in \Sigma$ for all $a \in A^\#$, $b \in B^\#$.

(2) $G$ satisfies the maximality condition (ascending chain condition) for $\Sigma$-subgroups.

If (1)(c) never occurs $\Sigma$ is called a degenerate class of $k$-root subgroups or a class of $k$-transvections of $G$. Otherwise $\Sigma$ is non-degenerate. Assume now that $G$ is generated by a class $\Sigma$ of $k$-root subgroups and $|k| > 3$ if $\Sigma$ is degenerate. Then it has been shown in [4] and [5] that there exists a nilpotent radical $R(G)$ satisfying

(a) $\bar{G} = G/R(G)$ is quasisimple,
(b) $\bar{R} = R(G)/R(G) \cap Z(G)$ is abelian,
(c) $C_{\bar{G}}(\bar{G}) = 0$ and $\bar{R} = [\bar{R}, \bar{G}]$,
(d) $[\bar{R}, \bar{A}, \bar{B}] = 0$ for all commuting elements $\bar{A}, \bar{B}$ of $\bar{\Sigma}$.

Moreover, in [4] simple groups generated by a class of $k$-transvections with $|k| > 3$ have been determined, the generic cases being symplectic groups over

$k$, respectively unitary groups over extension skew fields of $k$. Further, if $\Sigma$ is non-degenerate and $\tilde{R} \neq 0$ it has been shown in [5] that one of the following holds for $G^* = \bar{G}/Z(\bar{G})$:

(A) $G^* \cong \text{PSL}_n(k), n \geq 3$.
(B) $G^*$ is an orthogonal group of Witt-index = 3 defined over $k$.
(C) $G^*$ is of rank 2. (Here the rank is defined internally. It is shown in [5] that a simple group $G^*$ of rank 2 is either $\text{PSL}_3(k)$ or of type $G_2$.)

There are two main applications of the forthcoming classification of nearly simple groups generated by $k$-root subgroups I can see at the moment, namely:

(I) determination of subgroups generated by long-root subgroups of algebraic groups defined over $k$;
(II) quadratic action.

Since the class of long-root subgroups of an algebraic group over $k$ satisfies our conditions (1) and (2) above, such a subgroup (as in (I)) is generated by a 'set' of $k$-root subgroups and is thus by one of the main propositions of [5] mod some nilpotent normal subgroup a central product of groups generated by classes of $k$-root subgroups. Thus, if one not only wants to determine the 'nearly simple' subgroups generated by long-root subgroups, a determination of $\mathbb{Z}\bar{G}$-modules $\tilde{R}$ satisfying (c) and (d) above seems to be necessary.

In his work on quadratic pairs for $p > 5$, Thompson has shown (after roughly one-third of the proof) that there exists a class of quadratically acting elementary abelian $p$-subgroups satisfying (1) above, where $k$ is some $\text{GF}(p^n)$. It is my hope that one can prove something similar (perhaps only in more restricted situations) even in the infinite case. If now $G$ acts irreducibly, then $\tilde{R} = 0$. But if not, then $\tilde{R}$ might be different from 0. Thus if one is also interested in the determination of non-irreducible quadratically acting groups, again the question of determining $\mathbb{Z}\bar{G}$-modules $\tilde{R}$ satisfying (c) and (d) above arises.

Now in the finite case this is easily done using the representation theory of Lie-type groups in their natural characteristic. But the infinite case is more complicated, since one needs to construct the field action. For this purpose we prove:

**THEOREM 1.** Suppose $G$ is a quasisimple group generated by a class $\Sigma$ of $k$-root subgroups with $\bar{G} = G/Z(G) \cong \text{PSL}_n(k), n \geq 2$, or $\text{PSp}(2n, k), n \geq 2$, and $\Sigma$ is the class of root groups of transvections of $\bar{G}$. Assume that $V$ is a $\mathbb{Z}G$-module