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QUASI-SYMMETRIC 3-DESIGNS WITH TRIANGLE-FREE GRAPH

ABSTRACT. The following result is proved: Let $D$ be a quasi-symmetric 3-design with intersection numbers $x, y (0 \leq x < y < k)$. $D$ has no three distinct blocks such that any two of them intersect in $x$ points if and only if $D$ is a Hadamard 3-design, or $D$ has a parameter set $(v, k, \lambda)$ where $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2) + 1$, $k = \lambda^2 + 3\lambda + 2$ and $\lambda = 1, 2, \ldots$, or $D$ is a complement of one of these designs.

1. INTRODUCTION

A $t$-$(v, k, 2)$-design is a collection of $k$-subsets (called blocks) of a $v$-set (the elements of which are called points) such that any $t$-tuple of points occurs in exactly $\lambda$ blocks. In a $t$-design let $\lambda_i$ denote the number of blocks containing a given $i$-tuple, with $1 \leq i \leq t$. Then the following identity is satisfied:

$$\lambda_i \binom{k-i}{t-i} = \lambda \binom{v-i}{t-i}. \quad (1)$$

Let $D$ be a $t$-$(v, k, \lambda)$-design and let $p$ be a point of $D$. The collections $D_p = \{B \setminus \{p\} : B $ is a block of $D, p \in B\}$ and $D^p = \{C : C $ is a block of $D, p \notin C\}$, then $D_p$ and $D^p$ are $(t-1)$-$(v-1, k-1, 2)$ and $(t-1)$-$(v-1, k, \lambda - 1 - \lambda)$ designs and are respectively called the derived and the residual designs of $D$ (see [5]).

A $(t-1)$-design $E$ is said to be extendable if $E = D_p$ for some $t$-design $D$; also $D$ is called an extension of $E$.

A symmetric design is a $2$-$(v, k, \lambda)$-design such that $b = \lambda_0 = v$, $r = \lambda_1 = k$ and any two blocks intersect in $\lambda$ points. A $t$-design with two block intersection numbers is said to be quasi-symmetric. Here we consider a quasi-symmetric $3$-$(v, k, \lambda)$-design with block intersection numbers $x, y (0 \leq x < y < k)$. (In the case $y = k$, it is easily seen that $D$ is a repetition of copies of a symmetric design.) Cameron [4] showed that quasi-symmetric 3-designs with an intersection number $0$ are precisely an extension of symmetric designs and classified them into four types. Quasi-symmetric 3-designs in which one of the intersection numbers was $1$ were classified by Calderbank and Morton [2] and Pawale and Sane [7].

A quasi-symmetric block design (2-design) is called a proper quasi-symmetric design if both intersection numbers occur. The block graph $\Gamma$ of such a design $D$ is a graph whose vertices are blocks of $D$ with two distinct vertices adjacent if and only if the corresponding blocks intersect in $y$ points.

It is well known that $F$ and $\overline{F}$, the complement of $F$, are strongly regular graphs. Particularly interesting quasi-symmetric designs are those in which $\overline{F}$ has no triangles. It is easy to see that $\overline{F}$ has no triangles if and only if $D$ has no three distinct blocks such that any two of them intersect in $x$ points. Such designs were first studied by Baartmans and Shrikhande [1]. They showed that, for fixed intersection numbers $0$ and $y$, there are finitely many parametrically possible such designs. This result was extended by Shrikhande [9] for fixed intersection numbers $x$, $y$, $y > 1 + \sqrt{1 + 8x + 5x^2}$. In this paper we characterize quasi-symmetric 3-designs with the above property (Theorem 3.2), and show that these are precisely the Hadamard 3-designs or the 3-designs with $x = 0$ occurring in Cameron's classification of extensions of symmetric designs or the complement of these designs.

Section 2 contains preliminary results needed in Section 3. For basic definitions and results we refer to [5].

### 2. Preliminaries

In this section we list some results needed in Section 3. Let $D$ be a proper quasi-symmetric 2-design with a standard parameter set $(v, b, r, k, \lambda; x, y)$, where $x, y$ are block intersection numbers with $0 \leq x < y < k$. Let $\Gamma$ denote the block graph of $D$ where two vertices are adjacent if and only if the corresponding blocks intersect in $y$ points. Let $\overline{\Gamma}$ denote the complement of $\Gamma$.

The following result is well known.

**Lemma 2.1 [5].** Both $\Gamma$ and $\overline{\Gamma}$ are strongly regular graphs. If $\Gamma$ is connected then its parameter set $(n, a, c, d)$ is given by

$$
n = b, \quad a = \frac{k(r-1) - x(b-1)}{(y-x)},
$$

$$
c = \frac{(r-\lambda)(k-x)}{(y-x)} - \frac{(k-x)}{(y-x)},
$$

$$
d = a - \frac{(r-\lambda)(k-x)}{(y-x)^2} + \frac{(k-x)^2}{(y-x)^2},
$$

where $y-x$ divides $k-x$. Also if $\overline{\Gamma}$ is connected then the parameters $\overline{a}, \overline{c}, \overline{d}$ of $\overline{\Gamma}$ are obtained by interchanging $x$ and $y$ in the formulas of $a, c, d$ respectively.

**Lemma 2.2 [9].** The following are equivalent for a quasi-symmetric design $D$ with block graph $\Gamma(b, a, c, d)$: