1. Introduction

It is well known [1, (2.1)] that every oriented link in $\mathbb{R}^3$ is isotopic to a closed braid. If $\alpha$ is a braid, we denote its 'closure' (i.e. the corresponding oriented link) by $\tilde{\alpha}$. A theorem of Markov [1, (2.3)] gives necessary and sufficient conditions for two braids to have isotopic closures. Using this result, we construct complex-valued functions $w(\alpha)$ such that $w(\alpha) = w(\beta)$ whenever $\tilde{\alpha} = \tilde{\beta}$, thereby obtaining a set of invariants for oriented links.

More precisely, let $B_{n+1}$ be the Artin Braid group on $n+1$ strings with generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ satisfying the relations

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i < n) \]

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i - j| \geq 2). \]

In order for a function $w$ on the disjoint union of the $B_n$ to be constant on $\mathbb{R}^3$-isotopy classes of closed braids, it is necessary and sufficient that

(I) $w(\alpha y \alpha^{-1}) = w(\gamma)$ for all $\alpha, \gamma \in B_{n+1},$

(II) $w(\gamma \sigma_n^{\pm 1}) = w(\gamma)$ for all $\gamma \in B_{n+1}.$

We refer to the above conditions as Markov moves of types I and II, respectively. Condition (I) is rather pleasing to the algebraic eye, and suggests that one look at characters of $B_{n+1}$. The difficulties then center around (II).

Our approach involves the construction of class functions $\tau$ on the ascending union of the $B_n$ which satisfy

\[ \tau(\sigma_i^{-1}) = \tau(\sigma_i) \quad \text{and} \quad \tau(\gamma \sigma_n^{\pm 1}) = \tau(\gamma) \tau(\sigma_n). \]

We then get invariance under type (II) moves by setting

\[ w(\gamma) = \frac{\tau(\gamma)}{\tau(\sigma_n)^n} \quad \text{for} \ \gamma \in B_{n+1}. \]

The functions $\tau$ factor through the Burau representation $\beta$ (more precisely, through $\beta \oplus \overline{\beta}$ where $\overline{\cdot}$ is the automorphism of $Z[t, t^{-1}]$ sending $t$ to $t^{-1}$). Squier [10] discovered that $\beta$ leaves a Hermitian form invariant whose anti-symmetric part is defined over $Z[[t + t^{-1}]/2]$, the fixed subring of $\overline{\cdot}$. In this way, we obtain a homomorphism $\beta^*: B_n \to \text{Sp}(2n, Z[X])$. We then specialize $X$ to a finite field $F$ and apply the character of the metaplectic representation of $\text{Sp}(2n, F)$ to obtain the function $\tau.$


The present paper is divided into seven sections. In Section 2, we review presumably known facts about the finite metaplectic representation. We use these in Section 3 to obtain an interesting function $\tau$ on the finite symplectic group, which we normalize to link invariant in Section 5. As a by-product, we get an effective algorithm for the computation of the metaplectic character of $\text{Sp}(2n, q)$. In Section 4, we study a version of the Burau representation which leaves a symplectic form invariant. This form is derived from the unitary form over $\mathbb{Z}[i, t^{-1}]$ discovered by Squier. In Section 6 we study the relationship of the metaplectic invariant to the Alexander invariant, and in Section 7 we connect a special case with the Seifert form and the $V$ polynomial.

2. THE METAPLECTIC REPRESENTATION OF $\text{Sp}(2n, q)$.

In this section, we collect some information about the finite metaplectic representation. Much, but perhaps not all, of this is already known (e.g. [5], [6]).

Let $V$ be a $2n$-dimensional vector space over $F_q$, the field of $q$ elements, where $q = p^r$, $p$ odd. There is, up to equivalence, a unique non-singular symplectic form on $V$ which we denote by $[,]$. We let $\text{Sp}(V)$ be the group of non-singular linear transformations $g: V \to V$ for which $[g(v), g(v)] = [v, v]$. For $g \in \text{Sp}(V)$, we let $V_0(g) = \ker(1 - g)$ and $V_1(g) = \text{im}(1 - g)$.

\begin{equation} \label{eq:1} \tag{2-1} \text{Let } g \in \text{Sp}(V), \text{ then } V_0(g)^\perp = V_1(g). \end{equation}

\textbf{Proof.} Let $v_0 \in V_0(g)$ and $v \in V$. Then

\[ [v_0, v - g(v)] = [v_0, v] - [v_0, g(v)] = [v_0, v] - [g(v_0), g(v)] = 0. \]

Thus, $V_1(g) \subset V_0(g)^\perp$. But $\dim(V_1(g)) = \text{codim}(V_0(g)) = \dim(V_0(g)^\perp)$ because the form is non-singular. \hfill $\square$

We define $H = H(V)$, the Heisenberg group on $V$, to be the set of all pairs \{(v, \alpha) | v \in V, \alpha \in F_q \}$ with multiplication

\[ (v, \alpha) \cdot (u, \beta) = (v + u, \alpha + \beta + [v, u]). \]

It is easily checked that

\begin{enumerate}[\text{(a)}]
\item $H$ is a finite group of order $q^{2n+1}$,
\item $Z(H) = \{(0, \alpha) | \alpha \in F_q \}$ and $H/Z(H) \cong V$,
\item $\text{Sp}(V)$ acts on $H$ via $g(v, \alpha) = (g(v), \alpha)$, permuting the set of coset representatives $S = \{(v, 0) | v \in V \}$ for $Z(H)$ in $H$,
\item $(u, 0)^{-1}(v, 0)^{-1}(u, 0)(v, 0) = (0, 2[u, v])$ for all $u, v \in V$.
\end{enumerate}

We note that $H$ can be represented as the group of $(n + 1) \times (n + 1)$ strictly upper triangular matrices $U = \{ (u_{ij}) | u_{ij} = 0 \text{ for } 2 \leq i < j \leq n \}$. 