SIMPLEXES IN SPACES OF CONSTANT CURVATURE*

ABSTRACT. In hyperbolic, Euclidean and spherical n-space, we determine, for each positive number l, the largest interval of the form \( a_n(t) \leq l_{ij} \leq b_n(t) \) which guarantees the existence of an n-simplex \( p_1, p_2 \cdots p_{n+1} \) with edge-lengths \( p_ip_j = l_{ij} \). (In spherical geometry of curvature 1 the interval is empty unless \( l \leq 2 \arcsin \sqrt{(n+1)/2n} \).) The assertion that these intervals are as large as possible is justified because each of them allows certain degenerate simplexes. We determine explicitly all of these critical configurations.

1. INTRODUCTION

We define an \((n+1) \times (n+1)\) matrix \( L = (l_{ij}) \) to be allowable if \( l_{ii} = 0 \) and \( l_{ij} = l_{ji} > 0, \ i \neq j \). Our object is to determine whether the entries in a given allowable matrix can be realized as the distances \( l_{ij} = p_ip_j \) among \( n + 1 \) points \( p_1, p_2, \ldots, p_{n+1} \) in a suitable metric space. In [1] we considered the case in which the metric space is Euclidean n-space and proved two theorems. The first gave necessary and sufficient conditions for \( L \) to be realizable in terms of the eigenvalues of the related \( n \times n \) matrix \( S = (s_{ij}), \ s_{ij} = l_{in+1}^2 + l_{jn+1}^2 - l_{ij}^2. \) The second gave a useful sufficient condition by describing the largest sup norm ball of realizable allowable matrices centred at the allowable matrix \( L_1 = (1 - \delta_{ij}) \) corresponding to the unit n-simplex.

In this paper we generalize our results from Euclidean n-space to hyperbolic and spherical n-space. For convenience we fix the curvatures of these spaces at \(-1\) and \(+1\) respectively and accordingly restrict entries in \( L \) to be \( \leq \pi \) in the spherical case. A key observation in our earlier proof and applications of the second Euclidean theorem was that if \( L \) is allowable and \( k > 0 \) then \( kL \) is realizable if and only if \( kL \) is realizable. The absence of this similarity principle for non-Euclidean spaces has led us to a new inductive approach which is applicable in all three spaces of constant curvature.

2. QUALITATIVE STATEMENT OF RESULTS

An allowable matrix \( L \) is called realizable in a metric space \( X \) if there exist \( n + 1 \) points \( p_1, p_2, \ldots, p_{n+1} \) in \( X \) (a realization of \( L \)) such that the entries in \( L \) occur as distances among the \( p \)'s: \( l_{ij} = p_ip_j \). If \( X \) is Euclidean, spherical or a hyperbolic \( n \)-space, a realizable matrix \( L \) is called full-dimensional or

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degenerate according to whether its realizations are full-dimensional or lie in an \( m \)-dimensional subspace of \( X \) for some \( m < n \).

A natural metric for the space of allowable matrices is given by the sup norm distance

\[
d(L, L') = \| L - L' \|_\infty = \max_{1 \leq i,j \leq n+1} | l_{ij} - l'_{ij} | .
\]

We show in the corollary to Theorem 1 that, in this metric, the full-dimensional and the non-realizable matrices each comprise open sets with boundary equal to the degenerate matrices. The matrix \( L(\Sigma) \) of a full-dimensional simplex \( \Sigma \) is therefore the centre of an open sup norm ball \( B \) of maximal radius \( \rho(\Sigma) \) consisting entirely of full-dimensional matrices

\[
B = \{ L : d(L, L(\Sigma)) < \rho(\Sigma) \}
\]
and this ball is bounded by the sup norm sphere

\[
S = \{ L : d(L, L(\Sigma)) = \rho(\Sigma) \}
\]
consisting of realizable matrices and including some degenerate ones. The degenerate matrices in \( S \) are called critical matrices for \( \Sigma \) and their realizations are called critical configurations. Our main results, which follow in detail, concern the critical configurations for regular simplexes.

In the case where \( \Sigma \) is a regular simplex of edge-length \( \sigma \) we write \( \rho(\Sigma) = \rho(\sigma) \). If \( X \) is a Euclidean \( n \)-space or a hyperbolic \( n \)-space of curvature \(-1\), we show that there is, up to congruence, a unique critical configuration. It is \( (n-1) \)-dimensional and consists of two lower dimensional regular simplexes \( A \) and \( B \) of edge-length \( \sigma + \rho(\sigma) \) sharing a common centroid and lying in completely orthogonal flats. The distance between a vertex of \( A \) and a vertex of \( B \) is \( \sigma - \rho(\sigma) \) and the distribution of vertices, \( a \) in \( A \) and \( b \) in \( B \), is found to be \( a = [(n+1)/2], b = n+1-a \) by minimizing \( \rho(\sigma) \) subject to this constraint.

The story when \( X \) is a spherical \( n \)-space of curvature \( 1 \) is somewhat different. In this setting \( \pi \) is the largest edge-length that can be realized so that any allowable matrix with an entry greater than \( \pi \) is automatically non-realizable. Also, as we show in a part of Theorem 2, the circumradius \( R \) of a regular simplex of edge-length \( \sigma \) is given by \( \sin R = \sqrt{\sigma} \sin(\sigma/2) \). This means that the allowable matrix with all off-diagonal entries equal to \( \sigma \) is full-dimensional for \( \sigma < \sigma^* = 2 \arcsin\sqrt{(n+1)/2n} \), degenerate for \( \sigma = \sigma^* \) and non-realizable for \( \sigma > \sigma^* \). The distinctive features of the spherical case arise because of the fact that 'a regular simplex can grow too big to be realizable'. When \( \sigma < \pi/2 \), we find that the presence of the upper bound \( \sigma^* \) is not felt and