MEASURE RATIO OF EQUIDECOMPOSABLE SETS

ABSTRACT. Let \( n > 2 \). There are Lebesgue measurable sets \( A \) and \( B \) in \( \mathbb{R}^3 \) such that \( \lambda(B)/\lambda(A) = r \) and \( A \sim_n B \) if and only if \( 2/n < r < n/2 \).

Two sets \( A \) and \( B \) in \( \mathbb{R}^3 \) are said to be equidecomposable using \( n \) pieces if there is a partition \( \{A_1, A_2, \ldots, A_n\} \) of \( A \) and a partition \( \{B_1, B_2, \ldots, B_n\} \) of \( B \) such that \( A_i \) and \( B_i \) are isometric for each \( i \) in \( \{1, 2, \ldots, n\} \). In this case we write \( A \sim_n B \).

Banach and Tarski have shown that if \( A \) and \( B \) are bounded sets in \( \mathbb{R}^3 \), each having nonempty interior, then \( A \sim_n B \) for some \( n \) [1]. If \( A \) and \( B \) happen to be measurable, their measures then need not be equal. Indeed, Robinson has shown that if \( A \) is a deleted ball \( \{x \in \mathbb{R}^3: 0 < \|x\| < k\} \), and \( B \) is the union of two disjoint translates of \( A \), then \( A \sim_4 B \) [4]. Wagon observes the following generalization based on Robinson's method. Let \( n \) be even and at least 4. If \( A \) is a deleted ball, and \( B \) is the union of \( n/2 \) disjoint translates of \( A \), then \( A \sim_n B \) [5, p. 73]. These results use the Axiom of Choice, which we also assume throughout.

Let \( \lambda \) be Lebesgue measure in \( \mathbb{R}^3 \). When obtaining \( n/2 \) deleted balls from one as above, \( n \) pieces suffice regardless of the measure \( \lambda(A) \) of each deleted ball. It is the number of deleted balls comprising \( B \) or the ratio \( \lambda(B)/\lambda(A) \) which determines \( n \) in these constructions. Roughly speaking, measure may be doubled using four pieces, tripled using six pieces, and so on. We show (Theorem 1) that similar efficient constructions are possible when the number of pieces is odd. Indeed, for each \( n \geq 2 \) (\( n \) odd or even), there are sets \( A \) and \( B \) in \( \mathbb{R}^3 \) such that \( \lambda(B)/\lambda(A) = n/2 \) and \( A \sim_n B \).

It follows from a theorem of Laczkovich that \( n/2 \) is the largest possible value for the ratio \( \lambda(B)/\lambda(A) \) when \( A \sim_n B \) and \( n \geq 2 \) [3, Theorem 4]. We also prove this here (Theorem 2). Note that \( 2/n \) is then the smallest possible value for this ratio. In fact (Theorem 3), given \( n \geq 2 \), the range of possible values for \( \lambda(B)/\lambda(A) \) is exactly the closed interval \([2/n, n/2]\). These results extend trivially to any \( \mathbb{R}^k \) with \( k \geq 3 \).

The following lemma is due to Dekker [2, p. 67, Lemma (a)].

LEMMA 1 (Dekker). Let \( s \geq 1 \) and let \( g_0, \ldots, g_s \) be rotations about the z-axis in \( \mathbb{R}^3 \) such that no \( g_i \) is the identity or a halfturn. Let \( h \) be a rotation about the x-axis with angle \( \alpha \), where \( \cos \alpha \) is transcendental. Let \( k_1, \ldots, k_s \) be nonzero integers. Then \( g_0 h^{k_1} g_1 h^{k_2} \cdots h^{k_s} g_s \) is not the identity.
Since the conclusion holds in particular if \( g_s = g_0^{-1} \), it follows also that \( h^s g_1 h^{k_2} \cdots h^{k_s} \) is not the identity. This is the form used in Theorem 1 below. (The conclusion also holds if exactly one of the rotations \( g_0, g_s \) is the identity. Dekker's caution regarding halfturns is necessary, since if \( g \) is a halfturn, then \( (gh)^2 \) is the identity.)

**LEMMA 2.** Let \( G \) be a graph such that each component \( G_0 \) of \( G \) has exactly one cycle, or is acyclic and has an infinite path. Then there is a bijection \( s : E(G) \to V(G) \) such that for each edge \( e \) in \( E(G) \), the vertex \( s(e) \) is one of the two ends of \( e \).

**Proof.** If \( G_0 \) has exactly one cycle \( v_1 e_1 v_2 e_2 \cdots v_1 \) then define \( s(e_i) = v_i \) for each edge \( e_i \) in the cycle. For each edge \( e \) not in the cycle, define \( s(e) \) to be the end of \( e \) which is more distant from the cycle. If \( G_0 \) is acyclic with an infinite path \( v_1 e_1 v_2 e_2 \cdots \), define \( s(e_i) = v_i \) for each edge \( e_i \) in the path. Then, for each edge \( e \) not in the path, define \( s(e) \) to be the end of \( e \) which is more distant from the path. To get \( s \) on all of \( G \), apply the Axiom of Choice. \( \square \)

**THEOREM 1.** Let \( n \geq 2 \). There are bounded open sets \( A \) and \( B \) in \( \mathbb{R}^3 \) such that \( \lambda(A) = 2, \lambda(B) = n \), and \( A \sim_n B \).

**Proof.** It suffices to assume \( n \geq 3 \). Let \( r \) be a rotation of order \( n \). Let \( S_1, S_2, \ldots, S_n \) be pairwise disjoint deleted balls each with unit volume, arranged so that \( r(S_i) = S_{i+1} \) for each \( i \) in \( \{1, 2, \ldots, n-1\} \), and hence \( r(S_n) = S_1 \). Let \( A = S_1 \cup S_2 \) and let \( B = S_1 \cup S_2 \cup \cdots \cup S_n \). Then \( \lambda(A) = 2 \) and \( \lambda(B) = n \).

Let \( c_1 \) be the center of \( S_1 \), and let \( c_2 \) be the center of \( S_2 \). Let \( h \) be a rotation with axis \( c_1 c_2 \) and angle \( \alpha \), where \( \cos \alpha \) is transcendental. Define \( f_k = r^k h^k \) for each \( k \) in \( \{1, 2, \ldots, n\} \). Then
\[
f_k(S_1) = S_m = f_j(S_2) \quad \text{when } k + 1 = m = j + 2 \pmod{n}.
\]

Therefore, each point in \( B \) has two possible preimages in \( A \), one in \( S_1 \) and one in \( S_2 \). Let \( G \) be the infinite \( n \)-regular bipartite graph with vertex set \( V(G) = A \) and edge set \( E(G) = B \), where the edge \( b \) has ends \( a_1 \) and \( a_2 \) when these are its two possible preimages. (The set \( E(G) \) should be viewed as an abstract set identified with \( B \) in order that it be disjoint from \( V(G) \).)

Let \( G_0 \) be a connected component of \( G \). Since every vertex of \( G_0 \) has degree greater than 1, \( G_0 \) has a cycle or an infinite path. We claim that \( G_0 \) has at most one cycle. Suppose not. Then there is an edge \( b \) of \( G_0 \) such that \( G_0 \setminus b \) is connected and has a cycle. Let \( a_1 \) and \( a_2 \) be the ends of \( b \) which lie in \( S_1 \) and \( S_2 \) respectively. Let \( m \) be such that \( b \in S_m \). Then, from the definition of \( G \), we have \( f_k(a_1) = b = f_j(a_2) \) where \( j \) and \( k \) are determined by the equation