RIGIDITY AND POLARITY

I: Statics of Sheet Structures

Abstract. Any projective polarity in 3-space transforms a statically (or equivalently, infinitesimally) rigid bar-and-joint framework into a statically rigid hinged sheetwork — a set of plane-statically rigid sheets, joined in pairs along hinge lines. In the more general class of jointed sheetworks, which is closed under polarity, static rigidity is also preserved by the polarities. In particular, the class of infinitesimally (or statically) rigid polyhedra, built with joints at the vertices and bars triangulating the faces, is closed under polarity.

1. Introduction

Since the work of Cauchy ([4], [12]), mathematicians have studied the rigidity of convex polyhedra with various types of faces ([5], [6], [14]). As the results of Alexandrov made explicit ([1], [2]), the faces need not be single rigid objects, but could be formed from a flat set of triangles, provided all the vertices lie along the natural edges. In a recent paper on the infinitesimal (or equivalently, static) rigidity of polyhedra formed by bar-and-joint frameworks on the faces ([27]), we showed that the basic rigidity or instability of such polyhedra is not changed by the insertion or removal of joints along the edges where two faces meet, if certain natural bars are present, nor by the replacement of one plane-rigid framework in a face by a second plane-rigid framework on the same vertices. In effect, the static and infinitesimal mechanical properties depend only on the presence of plane-rigid faces, joined along the edges, and on the geometry of the chosen realization.

Other recent studies have emphasized the fact that the infinitesimal and static behaviour of a framework depends only on the projective geometry of the structure ([7], [8], [23], [24], [26], [28]). This was well known in the last century when geometers and engineers were not separated ([9], [16], [18], [19], [20]), but lost in the general neglect of geometry and the divisions among fields during this century. As part of a general program to apply modern combinatorial and geometric methods to the study of rigid structures ([3]), many of the standard constructions of projective geometry, such as projection, section and projective transformations, have been applied to rigid structures. With this background, we posed the extreme projective question — What happens when a structure is polarized? Are the basic static and mechanical properties preserved? After achieving a strange,
but intriguing, interpretation for the polars of plane structures ([29], we
pursued this elusive possibility in space for several years.

In time these two strands – the plane-rigid sheets and the projective
polarities – wove together into a single theory of sheet structures. In 3-space
the projective polar of a point is a plane, or a section of a plane. If each
plane section is built as a plane-rigid bar-and-joint framework, with suitable
joints, and these sheets are hinged together along selected lines of in-
tersection, this hinged sheetwork is a candidate for the polar, or projective
dual of a bar-and-joint framework, built in space with universal joints and
rigid bars. With the ‘natural’ definitions of infinitesimal mechanical and
static properties for such sheet structures, the desired goal is attained: the
spatial polarity preserves all static and infinitesimal properties, such as
rigidity, degrees of freedom, etc.

In this paper we shall present the statics of this polarity (Sections 2 and
3). For reasons of space, we delay the infinitesimal mechanics of the polarity
to a second paper. The choice of statics is based on the simple algebraic and
geometric pattern assumed by equilibrium loads and their resolutions when
these forces are translated into projective line geometry.

Generalizing the geometry and the algebra of this polarity, we define a
new self-polar class of structures – the jointed sheetworks (Section 4). Such
a structure is formed of plane-rigid sheets contacting other sheets at
designated joints. Under spatial polarities the joints become plane sheets
and the sheets become joints of contact – and the polarity preserves all
static (and infinitesimal mechanical) properties. Such sheetworks are mo-
deled by taking all the contact points in a sheet and building a plane-
statically rigid bar-and-joint framework on these points. For non-
degenerate sheets we show that this model will be statically rigid if and only
if the jointed sheetwork is. We also show that a sheet with exactly two joints
functions like a bar, so that bar-and-joint frameworks, and their polars, are
special examples of jointed sheetworks.

In Section 5 we turn to the classical polyhedra built as jointed sheet-
works. This leads to the basic theorem that a polyhedron, built with joints
at the vertices and along the natural edges, and plane-rigid bar-and-joint
faces is statically rigid if and only if the dual or polar polyhedron is also
statically rigid when similarly built. As we mentioned above, it has been
known for over a century that infinitesimal and static rigidity are invariant
under projective transformations, but this appears to be the first result on
invariance under polarity (see Section 7).

The hinged sheetworks are the projective polars of bar-and-joint frame-
works. These bar-and-joint frameworks have been extended to tensegrity