Abstract. It is shown that isometries between the unit spheres of finite dimensional Banach spaces necessarily map antipodal points to antipodal points.

Let $M$ and $M'$ be finite dimensional Banach spaces with unit spheres $S$ and $S'$ and unit balls $B$ and $B'$. If the possibility of confusion exists we use $\| \cdot \|_S$ and $\| \cdot \|_S'$ for the norms, otherwise we simply use $\| \cdot \|$.

Suppose that $f: M \rightarrow M'$ is an isometry, meaning that $\|x - y\|_S = \|f(x) - f(y)\|_{S'}$ for all $x, y$ in $M$. In [4] Mazur and Ulam showed that $f$ is necessarily an invertible affine transformation (that is, a linear transformation composed with a translation) of $M$ onto $M'$. In [3] Mankiewicz extended this, showing that if $U \subset M$ and $V \subset M'$ are either open and connected or convex bodies and $f: U \rightarrow V$ is an isometry then $f$ is the restriction to $U$ of an affine transformation of $M$ onto $M'$. (Both of the above quoted results were proven for the infinite as well as the finite dimensional case.) Applying these results to the unit balls of $M$ and $M'$ it follows that $M$ and $M'$ are linearly isometric if and only if their unit balls are (metrically) isometric. These results seem natural for one feels that isometries 'should' be linear and that the unit ball 'determines' the space.

On the other hand, there are subsets of Banach spaces which are isometric, but not in any sense affinely isometric. Consider the mapping $f: \mathbb{R} \rightarrow l^\infty_2$ ($\mathbb{R}^2$ with the max norm) given by

$$f(x) = \begin{cases} (x, x), & x \geq 0 \\ (x, -x), & x < 0. \end{cases}$$

Then $f$ is an isometry, but is clearly not affine. Perhaps the reason for this is that the range of $f$ does not contain sufficient directions of the space.

A very natural set to consider which one intuitively feels determines the space is the unit sphere. Thus, suppose that $f: S \rightarrow S'$ is an isometry. Is $f$ necessarily the restriction to $S$ of a linear, or affine, transformation? We are unable to answer this question, even in two dimensions.

One approach to this problem is to define the extension of $f$, call it $T$, and to show that $T$ is affine. The function $T$ may be defined as follows: For $x \neq 0$ in $M$, $T(x) = \|x\| \cdot f(x/\|x\|)$ and $T(0) = 0$. It is immediate that $T$ is positively homogeneous (that is, $T(\lambda x) = \lambda T(x)$ for $\lambda > 0$). Thus a first step in showing that $T$ is linear would be to show that $T$ is homogeneous. In effect, this is what we show here. Specifically, we show that if $M$ and $M'$ are finite dimensional Banach spaces with spheres $S$ and $S'$ and $f: S \rightarrow S'$ is an
isometry, then $f(-x) = -f(x)$ for all $x$ in $S$. Note that this can be interpreted as showing that antipodal points are determined by the intrinsic metric of the sphere.

Although we feel this result is interesting in its own right, we hope that it will serve as a stepping stone (at least for finite dimensions) to showing that $f$ is the restriction of a linear function.

1. Preliminaries

Much of the terminology used here is standard, and can be found in any book on convexity. We refer the reader to any of [1], [2], or [5]. Here we list the notation used as well as the less standard terminology.

Let $A \subset \mathbb{R}^n$. Then $\text{aff } A$ will be the affine hull of $A$, $\text{conv } A$ the convex hull of $A$, $\text{ext } A$ the set of extreme points of $A$, and $\text{relint } A$ the relative interior of $A$. It is well known that if $K \subset \mathbb{R}^n$ is closed and convex, then $K$ is the closure of $\text{relint } K$ [1, p. 16]. If $p$ and $q$ are in $\mathbb{R}^n$, then $[p, q]$ or $[pq]$ is the segment from $p$ to $q$ and $pq$ the line through $p$ and $q$. If $p$, $q$, and $r$ are distinct and $q \in [pr]$, we write $[pqr]$.

A face of a closed convex set $K \subset \mathbb{R}^n$ is $K \cap H$ where $H$ is a hyperplane supporting $K$. A facet of $K$ is a maximal face, maximal in the sense that it is not properly contained in any other face. It is easy to show that each face of $K$ and each point of the boundary of $K$ is contained in a facet. We shall be dealing with faces, facets, extreme and exposed points of $B$, the unit ball of a norm on $\mathbb{R}^n$. We also refer to these as faces, facets, extreme and exposed points of $S$, the unit sphere.

If $A \subset \mathbb{R}^n$, the star of $x$ with respect to $A$, $\text{st}(x, A)$, is defined by

$$\text{st}(x, A) = \{y \in A : [xy] \in A\}.$$ 

2. Characterizations of Facets

Let $S$ be the unit sphere of a finite dimensional Banach space. This section contains a series of lemmas that characterize facets and stars of $S$ in terms of the metric.

**Lemma 1.** A set $F$ is a facet of $S$ if and only if $F$ is a maximal convex subset of $S$.

**Proof.** Since a facet is a face, it is convex. Suppose that $F$ is a facet of $S$ and $F \subset K \subset S$, where $K \subset S$ is convex. Since $\text{relint } K \subset S$, then $\text{relint } K \cap \text{relint } B = \emptyset$ (where $B$ is the unit ball) so $K$ and $B$ can be separated by a hyperplane $H$ [2, p. 11, Th. 2]. (That is, $K$ and $B$ lie in different closed half