ABSTRACT. Let $t > r$ be an integer. If $G$ is a group acting flag-transitively on a finite linear space and $G^0$ is a normal subgroup of $G$ with $t$ orbits on the flags, then $G^0$ is point-primitive up to a finite number of exceptions.

1. INTRODUCTION

Let $S$ be a finite linear space or Steiner system $S(2, k, v)$ with $k < v$ and $G$ a group of automorphisms of $S$ acting flag-transitively on $S$. Then the theorem of Higman–McLaughlin ([5] or [1]) asserts that $G$ acts primitively on the points. The proof of this important result is using very little of the geometry and of the group theory; it is essentially combinatorial. It calls for generalizations but none is known. One of our goals here is to use the same methods in a more general setting. Assume that $G^0 \neq 1$ is a normal subgroup of $G$. Then $G^0$ is no longer necessarily primitive. It suffices to think of the case where $S$ is an affine space and $G^0$ is the translation group. However, other known examples indicate that $G^0$ is usually flag-transitive, hence primitive. We shall prove a generalized Higman–McLaughlin theorem whose purpose is to put very strong restrictions on $S$ when $G^0$ is imprimitive and of index $\leq 3$ in $G$. The result can be applied to the classification program [2]. The author thanks A. Pasini for helpful remarks on the first version of this paper.

2. REDUCTION TO ARITHMETIC

2.1. Let $S$ be a finite linear space of $v$ points, with lines of $k$ points, $2 \leq k < v$. Then the number of lines on a point is

$$r = \frac{v - 1}{k - 1}$$

and the number of lines is

$$b = \frac{v(v - 1)}{k(k - 1)}.$$
Moreover,

\[ k \leq r \quad \text{and} \quad k^2 - k + 1 \leq v \quad (\text{see [1]}). \]

2.2. Let \( G \) be a group of automorphisms of \( S \) acting transitively on the flags of \( S \), i.e. the incident point-line pairs.

Let \( G^0 \neq 1 \) be a normal subgroup of \( G \) and \( B \) a partition of the point set of \( S \) in \( n \) blocks of \( c \) points such that \( B \) is \( G^0 \)-invariant. Assume \( c \geq 2, n \geq 2 \).

Hence

\[ v = cn. \]

Let \( t \) be the number of orbits of \( G^0 \) on the flags of \( S \). Since \( G \) is point-primitive, \( G^0 \) is point-transitive and so, for a given point \( p \), the stabilizer \( G^0_p \) has \( t \) orbits of length \( r/t \) on the lines through \( p \).

2.3. Let \( p \) be a point and \( B \) the block on \( p \). For each orbit of \( G^0_p \) on the lines through \( p \), let \( l_i, i = 1, \ldots, t, \) be a representative line. Let \( d_i = |B \cap l_i| \). Then

\[ c - 1 = \frac{r}{t} \sum_{i=1}^{t} (d_i - 1). \]

Put \( d = \sum_{i=1}^{t} (d_i - 1) \). Since \( c \geq 2 \), we get \( d \geq 1 \). Hence we reach the following conclusion.

2.4. PROPOSITION. Under the assumptions made in 2.1 and 2.2 there is an integer \( d > 1 \) such that

\[ r = \frac{t(c - 1)}{d} \quad \text{and} \quad d | c - 1. \]

2.5. Using (1) to (5) and the fact that \( k, v, r, b, c, n, t \) and \( d \) are integers with \( v > k > 2, r > v, c > 2, n > 2, t > 1 \) we can now derive rather strong constraints by a purely arithmetical analysis.

3. THE ARITHMETICAL ANALYSIS

3.1. Let us start with integers \( k, v, r, b, c, n, t, d \) such as in 2.5.

3.2. There exists an integer \( e \) such that \( e \geq 1 \) and

\[ n = \left( \frac{e(c - 1)}{d} \right) + 1. \]

Proof. By (5), \( c - 1 | dr \) so by (1), \( c - 1 | d(v - 1) \), hence \( c - 1 | (nc - 1)d \) by (4), etc.