CHARACTERISTIC CLASSES OF ALMOST-FLAG STRUCTURES

Abstract. The aim of this paper is to present a basic theory of characteristic classes of almost-flag structures. Vanishing theorems for primary characteristic classes of almost-flag and almost-product structures are proved. Consequences of these theorems for secondary characteristic classes are drawn in the framework of Lehmann's theory. Two theorems on residues of flag structures complete the paper.

1. Introduction

The aim of this paper is to present a basic theory of characteristic classes of almost-flag structures. Our treatment follows Chern–Bott's method; that is, by the characteristic classes of an almost-flag structure we understand the characteristic classes of the corresponding reduction of the normal bundle of the smallest distribution of the structure. Moreover, we prove our theorems by constructing special connections, as has been done previously by Bott and Martinet. In the paper we prove vanishing theorems for primary characteristic classes of almost-flag structures and almost-product structures. Then we draw conclusions of these theorems for secondary characteristic classes in the framework of Lehmann's theory. Finally, we present two theorems on residues of flag structures, which are generalizations of Heitsch's theorems for foliations to the case of flag structures (cf. [4]).

The vanishing theorem for foliations has been proved by Bott in 1969 (cf. [1]), distributions by Martinet in 1972 (cf. [7]), flag structures by Cordero and Masa in 1980 (cf. [3]) and almost-flag, almost-product structures by the author independently in 1980 (cf. [9]). The proof presented in this paper is an improved version of the one presented in [9]. The two theorems on residues have been announced in [9].

All the manifolds, vector fields, differential forms, etc., considered in this paper are smooth, i.e. differentiable of class $C^\infty$. If the manifold is denoted by $M$, $TM$ denotes the tangent bundle, $\mathcal{X}(M) = \Gamma(TM)$ the space of global sections of $TM$, i.e. the set of global vector fields on $M$, which is a Lie algebra.

A subbundle of the tangent bundle $TM$ we call a distribution. An almost-flag structure is a finite set of distributions $F = (F_1, \ldots, F_k)$ ordered by inclusion, i.e. $F_i \subseteq F_{i+1}$. The set $F$ is a flag structure if all the distributions of $F$ are involutive. An almost-product structure is a set of distributions $F = (F_1, \ldots, F_k)$ such that $\bigoplus F_i = TM$. To get an integrable structure one needs more than mere involutions of each distribution $F_i$. When the $G$-
structure defined by \( F \) is integrable, the almost-product structure \( F \) is called a product structure. A Riemannian metric \( g \) is said to be compatible with a flag structure \( F = (F_1, \ldots, F_k) \) if for any \( i = 1, \ldots, k - 1 \), \( F_i \oplus F_i^- \cap F_{i+1} = F_{i+1} \). It is straightforward to show that such a metric exists. For a more complicated case, see [2].

1. PRIMARY CHARACTERISTIC CLASSES

Let \( F = (F_1, \ldots, F_k) \) be an almost-flag structure, \( \operatorname{codim} F_i = q_i \). Let \( G(q) = \operatorname{GL}(q_1; q_1 - q_k, \ldots, q_1 - q_2) \) denote the Lie group of \( q_1 \times q_1 \) matrices leaving the following subspaces of \( \mathbb{R}^{q_1} \) invariant:

\[
V_i = \{ x \in \mathbb{R}^{q_1} : x_s = 0 \quad \text{for} \quad s > q_1 - q_i \}.
\]

By the characteristic classes of an almost-flag structure \( F \) we understand the characteristic classes of the reduction to the group \( G(q) \) of the bundle of the linear frames of the vector bundle \( TM/F_1 \). Therefore it is necessary to know the structure of the algebra \( I(G(q)) \) of invariant symmetric multilinear mappings on \( q(q) \) – the Lie algebra of the group \( G(q) \).

**Proposition 1** [9]. The algebra \( I(G(q)) \) is isomorphic to the tensor algebra \( I(GL(q_k)) \otimes I(GL(q_{k-1} - q_k)) \otimes \cdots \otimes I(GL(q_1 - q_2)) \).

**Notation.** Let \( c_{ij} \in \bigotimes_{s=1}^{k} I(GL(q_s - q_{s+1})) \) be the mapping \( 1 \otimes c_j \otimes 1 \), where \( c_j \) is on the \( i \)th place; the inverse image by the isomorphism of \( c_{ij} \) is denoted by \( c^i_j \).

To formulate a vanishing theorem for almost-flag structures we need the following definition.

An almost-flag structure \( F = (F_1, \ldots, F_k) \) is of type \( (r_1, \ldots, r_k) \) if \( r_i = \sup_{m \in M} \{$\\operatorname{codim} C_i m$\} \), where \( C_i \) is the distribution generated by the sheaf \( F_i \cap U_f F_i \) is the sheaf of germs of sections of the distribution \( F_i \) and \( U_f \) is the sheaf of germs of infinitesimal automorphisms of the almost-flag structure \( F \).

Now we shall construct the set of connections which will be called \( F \)-connections. Assume that a family \( S = \{S_1, \ldots, S_k\} \) of subbundles of \( TM \) having the following properties is given:

1. \( F_i \oplus S_i = F_{i-1} \) for \( i = 1, \ldots, k - 1 \),
2. \( F_k \oplus S_k = TM \).

Then the bundle \( S_i \) is isomorphic to the bundle \( F_{i+1}/F_i \) and the bundle \( TM/F_1 \) to the bundle \( S_1 \oplus \cdots \oplus S_k \). In the vector bundle \( S^i = \bigoplus_{j>i} S_j \) we define a connection \( \nabla^i \):