ON ROSETTES AND ALMOST ROSETTES

Abstract. The closed plane curves of class $C^2$ which have curvature $k(s) > 0$ or $k(s) \geq 0$ with a finite number of zeros are studied. The results concern the existence of normal lines which divide the perimeter into equal parts and the existence of some special kinds of pairs of points on these curves as orthodiameter pairs, antipodal pairs, etc. The paper also contains some generalizations of the theorems of Blaschke-Süss and Barbier.

1. Preliminaries

In this paper we shall consider the class of all positively oriented rosettes and almost rosettes. We recall that $C^2$, a plane closed curve of positive curvature, is called a rosette (see [2]). Ovals (see [3]) are a special class of rosettes.

By $s, L$ and $k$ we shall denote, respectively, the arc length, the perimeter and the curvature of a fixed rosette.

Let us consider a rosette $C, s \rightarrow z(s) = x(s) + iy(s)$ for $0 \leq s \leq L$. The tangent and normal vectors to $C$ at $z(s)$ are denoted by $T(s)$ and $N(s)$, respectively. We shall consider the functions $z, T, N, k$ to be periodic of the period $L$. If $a$ is the smallest number such that $0 < a < L$ and $T(a) = e^{i\phi} T(0)$, then the solution of the differential equation (see [2])

$$\varphi' = \frac{k}{k \varphi}$$

with the initial condition

$$\varphi(0) = a$$

satisfies the relation

$$T(\varphi(s)) = e^{i\varphi} T(s) \quad \text{for } s \in \mathbb{R}.$$ 

By $\psi$ we denote the solution of (1) such that $T \circ \psi = -T$.

Let us consider a vector and functions:

$$p = z - z \circ \psi, \quad \delta = -\langle p, N \rangle, \quad \Delta = \langle p, T \rangle, \quad 2\iota = \frac{1}{k} + \frac{1}{k \circ \psi},$$

where $\langle , \rangle$ denotes the Euclidean scalar product in the plane, $\delta$ is the width function and $\iota$ is the mean radius of curvature.

2. Orthodiameter pairs

Definition 1[2]. A pair of points of a rosette which lie on the same normal line shall be called an orthodiameter pair.

The existence of such a pair is shown in [2].

**THEOREM 1.** Each rosette has at least two orthodiameter pairs.

*Proof.* Let \(|p|^2 = \langle p, p \rangle\). Then we have \(|p'| = (1/|p|)kz\Delta\). As the function \(s \mapsto |p(s)|\) is defined and continuous on a compact set, it reaches its maximum and minimum. These extremes determine the orthodiameter pairs.

### 3. Integral Formulas

Let \(C\) be a rosette, \(s \mapsto z(s)\) for \(0 \leq s \leq L\), and let us recall the notion of an \(L\)-involution.

**DEFINITION 2[1].** Each function \(v: [0, +\infty) \to \mathbb{R}\) which satisfies the conditions

\[
\begin{align*}
0 < v(0) < L \\
v'(a) > 0 & \quad \text{for all } a > 0, \\
v(a + L) = v(a) + L & \quad \text{for all } a \geq 0, \\
v(v(a)) = s + L & \quad \text{for all } a \geq 0
\end{align*}
\]

will be called an \(L\)-involution.

Let \(v\) be a fixed \(L\)-involution and let us introduce a vector and some functions:

\[
\begin{align*}
q &= z - z^o v, & \omega &= -\langle q, N \rangle \\
\Omega &= \langle q, T \rangle, & \gamma &= K^o v - K,
\end{align*}
\]

where \(K(s) = \int_0^s k(r)dr\). Let us note that

\[
\gamma^o v = -\gamma + \omega j,
\]

where \(j\) denotes the index of \(C\).

It can be derived from (6) that

\[
\omega' = k\Omega + v' \sin \gamma, \quad \Omega' = -k\omega + 1 - v' \cos \gamma.
\]

Integrating the above system of differential equations we get

**THEOREM 2.** For a rosette \(C\) and an arbitrary \(L\)-involution \(v\) the following integral formulas hold:

\[
\begin{align*}
\int k(s)\Omega(s) \, ds - \int \sin \gamma(s) \, ds &= 0 \\
\int k(s)\omega(s) \, ds + \int \cos \gamma(s) \, ds &= L.
\end{align*}
\]

We are now able to show the existence of special normal lines.