ON TRANSVERSAL INFINITESIMAL
AUTOMORPHISMS FOR HARMONIC FOLIATIONS

ABSTRACT. In this paper we consider a harmonic Riemannian foliation $\mathcal{F}$, and study the
transversal infinitesimal automorphisms of $\mathcal{F}$ with certain additional properties like being
transversal conformal or Killing (=metric). Such automorphisms (modulo Killing automor-
phisms) are related to the stability of $\mathcal{F}$. A special study is made for the case of a foliation with
constant transversal scalar curvature, and more particularly with transversal Ricci curvature
proportional to the transversal metric (Einstein foliation).

1. INTRODUCTION

Let $\mathcal{F}$ be a transversally oriented foliation on a compact oriented
Riemannian manifold $(M, g_M)$. It is given by an exact sequence of vector
bundles

$$0 \to L \to TM \overset{\pi}{\to} Q \to 0,$$

where $L$ is the tangent bundle and $Q$ the normal bundle of $\mathcal{F}$. We have an
associated exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathcal{F}) \overset{\pi}{\to} \Gamma Q^2 \to 0,$$

where $V(\mathcal{F})$ denotes the algebra of infinitesimal automorphisms of $\mathcal{F}$, and
$\Gamma Q^2$ the portion of $\Gamma Q$ invariant under the action of $L$, by Lie derivatives
([5], [12]). The foliation is assumed to be Riemannian with bundle-like
metric $g_M$, and holonomy invariant induced metric $g_Q$ on $Q \cong L^\perp$. The
unique metric and torsion-free connection in $Q$ is denoted by $\nabla$ ([4], [12]).
Associated to $\nabla$ are transversal curvature data, in particular, the (transver-
sal) Ricci operator $\rho_\nabla: Q \to Q$ and the (transversal) scalar curvature $c_\nabla = \text{trace } \rho_\nabla$ ([5]). In this paper we study geometric properties of infinitesimal
automorphisms $Y \in V(\mathcal{F})$. For $Y \in V(\mathcal{F})$ the transversal part $\pi(Y)$ of $Y$ is also
denoted by $\tilde{Y}$ and $\omega$ will stand for the basic 1-form associated to $\tilde{Y}$ by $(g_Q)$
duality.

Recall that the basic forms are given by

$$\Omega^*_g = \{ \omega \in \Omega^*_M : i(X)\omega = 0, \Theta(X)\omega = 0 \text{ for all } X \in \Gamma L \}.$$

The exterior differential $d$ restricts to $d_g: \Omega^*_g \to \Omega^{*+1}_g$. The adjoint of $d_g$, with

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With respect to the induced scalar product \( \langle , \rangle_B \) on \( \Omega_B \), is denoted by \( \delta_B : \Omega_B^* \to \Omega_B^{-1} \) and we then have the (basic) Laplacian \( \Delta_B = \delta_B d_B + d_B \delta_B \). The relation expressing \( \delta_B \) in terms of \( \nabla \) involves the mean curvature form of \( \mathcal{F} \) which, in this paper, we assume to be zero. In other words, \( \mathcal{F} \) is assumed to be harmonic, i.e. all leaves of \( \mathcal{F} \) are minimal ([4]).

By [8], [9], the De Rham–Hodge decomposition generalizes to a decomposition

\[
\Omega_B^* \cong \text{im } d_B \oplus \text{im } \delta_B \oplus \mathcal{H}_B^*,
\]

into mutually orthogonal subspaces, with finite dimensional space of harmonic basic forms \( \mathcal{H}_B^* = \ker \Delta_B \).

In [10] the operators \( \delta^*, \delta \) occurring in the Berger–Ebin decomposition [2] were generalized to the foliation context.

\[
\delta^* : \Gamma Q^* \to \Gamma S^2 Q^*, \quad S^2 = \text{symmetric square},
\]

is given by

\[
(\delta^* \omega) (V, W) = \frac{1}{2} \left\{ (\nabla_V \omega) (W) + \nabla_W \omega (V) \right\}, \quad \omega \in \Gamma Q^*, \quad V, W \in \Gamma Q.
\]

It maps the basic 1-forms \( \Omega_B^* \subset \Gamma Q^* \) to basic symmetric 2-forms, i.e. those killed by \( \iota(X), \Theta(X) \) for all \( X \in \Gamma L \). For the present purpose it suffices to know that \( \delta : \Gamma S^2 Q^* \to \Gamma Q^* \) defined in [10] restricts on basic forms to the adjoint of \( \delta^* \). The fundamental identities for \( Y \in V(\mathcal{F}) \) and \( \omega = g_Q \)-dual of \( Y \), in the case of a harmonic Riemannian foliation \( \mathcal{F} \), are then ([10])

\[
\begin{align*}
(1.1) \quad & 2 \delta \delta^* \omega = - \text{trace } \nabla^2 \omega - \rho_V (\omega) + d_B \delta_B \omega, \\
(1.2) \quad & \text{div}_B \tilde{Y} = - \delta_B \omega = (\delta^* \omega, g_Q), \\
(1.3) \quad & |\delta^* \omega - \frac{1}{q} \text{div}_B \tilde{Y} \cdot g_Q|^2 = |\delta^* \omega|^2 - \frac{1}{q} (\text{div}_B \tilde{Y})^2, \quad q = \text{codim } \mathcal{F}.
\end{align*}
\]

Let \( Y \in V(\mathcal{F}) \). Then \( \tilde{Y} \) is divergence free if \( \text{div}_B \tilde{Y} = 0 \), a transversal Jacobi automorphism if \( Y \in \ker J_V \), where \( J_V = - \text{trace } \nabla^2 - \rho_V \) is the Jacobi operator, a transversal Killing automorphism if \( \Theta(Y) g_Q = 0 \) and transversal conformal if \( \Theta(Y) g_Q = \mu \cdot g_Q \) for some basic function \( \mu \). These properties can be equivalently expressed in terms of the \( g_Q \)-dual \( \omega \) by \( \delta_B \omega = 0 \), trace \( \nabla^2 \omega + \rho_V (\omega) = 0, \delta^* \omega = 0 \) and \( \delta^* \omega = -(1/q) \delta_B \omega \cdot g_Q \), respectively ([10]).

This motivates the introduction of the following concept.

\[\text{2.\sigma-Automorphisms}\]

Given \( Y \in V(\mathcal{F}) \), \( \tilde{Y} \) is said to be a \( \sigma \)-automorphism for \( \sigma \in \mathbb{R} \) if (the \( g_Q \)-dual)\( \omega \) satisfies

\[
(2.1) \quad - \text{trace } \nabla^2 \omega - \rho_V (\omega) + \sigma d_B \delta_B \omega = 0.
\]