ON A COMBINATORIAL PROBLEM CONCERNING
SUBPOLYTOPES OF STACK POLYTOPES

Abstract. Shephard [6] posed the problem of whether every combinatorial type of
d-polytope can be represented by some subpolytope of some d-stack. Here we show that,
at least in the case $d = 3$, the answer to that problem is in the affirmative.

I

Let $\mathbb{P}^d$ denote the set of all $d$-dimensional convex polytopes ($d$-polytopes) in
dimensional Euclidean space $\mathbb{R}^d$. For explanations of standard terminology
and notations used here, see [2].

Definition 1. Let $P \in \mathbb{P}^d$, $F$ be a facet of $P$ and $v$ be a point beyond $F$
beneath all the other facets of $P$. Then the polytope $P' = \text{conv}(\{v\} \cup F)$
is said to arise from $P$ by adjoining the pyramid $\text{conv}(\{v\} \cup F)$ to the facet $F$
of $P$.

Definition 2. Let $\{S_1, \ldots, S_n\}$ be a set of $d$-polytopes and $S_1$ be a $d$-
simplex. Then $S_n$ is called a $d$-stack (of $n$ components) if either $n = 1$ or
$n \geq 2$ and $S_{i+1}$ arises from $S_i$ by adjoining a simplex to one of the facets of $S_i$
($i = 1, \ldots, n - 1$). (The simplexes that are adjoined together with $S_1$ are
called the components of $S_n$.)

As in [6] we denote by $\mathbb{S}^d$ the class of $d$-stacks. It follows from Definition 2
that for any $S \in \mathbb{S}^d$ the graph $G(S)$ of $S$ is either a complete graph of $d + 1$
nodes or the union of complete graphs of $d + 1$ nodes. Necessary and sufficient conditions for a (finite) graph being isomorphic to the graph of a $d$-stack
were given by Kleinschmidt [3].

Definition 3. Let $P, Q \in \mathbb{P}^d$. We call $P$ a subpolytope of $Q$ if $\text{vert } P \subseteq
\text{vert } Q$ and write $P \leq Q$.

In [6], Shephard asked the question of whether every polytope is a sub-
polytope of some stack polytope. He gave a negative answer by constructing
a counter-example. As that example depends only on metric properties of the
polytope, he posed the problem of whether at least every combinatorial type
of $d$-polytope can be represented by some subpolytope of some $d$-stack. We

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give here a partial solution to that problem; namely, in the case \( d = 3 \). We shall prove:

**THEOREM.** Every combinatorial type of 3-polytope can be represented by some subpolytope of some 3-stack.

**II**

The proof of the theorem makes use of operations on graphs of 3-polytopes which are described in [1], [4], [5] and [7]. We call a graph 3-polyhedral if it is isomorphic to the graph of some 3-polytope. The famous theorem of Steinitz (see [1], [2], [4], [7]) states that a graph is 3-polyhedral if and only if it is planar, 3-connected and has no multiple edges. Therefore, we may assume that all 3-polyhedral graphs are embedded in the plane.

Using graph-theoretic terms, our theorem reads:

*Every 3-polyhedral graph is realizable by a subpolytope of some 3-stack.*

Our proof depends on the following result by Kirkman [see Steinitz and Rademacher [7], p. 219]:

**LEMMA.** Any 3-polyhedral graph with \( e + 1 \) edges that is not isomorphic to the graph of a pyramid can be obtained from some 3-polyhedral graph with \( e \) edges by one of the following two (dual) operations (see [1], [4] and [7]):

(a) Face splitting of the first kind

(b) Vertex splitting of the first kind

For pyramids our theorem is easy to prove. Let \( B_{n+1} = \text{conv}\{x_0, x_1, \ldots, x_n\} \) be a regular \( (n + 1) \)-gon in a plane \( \pi \) \((n \geq 3)\) and let \( I = \text{conv}\{y, z\} \) be a segment orthogonal to \( \pi \) such that \( x_0 \) is the midpoint of \( I \). Then \( \text{conv}\{y, x_1, \ldots, x_n, z\} \) is obviously a 3-stack and \( \text{conv}\{y, x_1, \ldots, x_n\} \) is a pyramid with basis \( B_n = \text{conv}\{x_1, \ldots, x_n\} \).

Now let \( P \) be a 3-polytope which is not a pyramid. Suppose \( P \) has \( e \) edges and let \( G(P) \) be the graph of \( P \). According to our lemma there exists a graph \( G(P') \) with \( e - 1 \) edges such that \( G(P) \) can be constructed from \( G(P') \) by operation (a) or (b).

Let us assume that our theorem is true for \( G(P') \), i.e. there exist \( P' \in \mathbf{P}^3 \) with graph \( G(P') \) and \( S(P') \in \mathbf{S}^3 \) such that

\[
P' \leq S(P') \in \mathbf{S}^3.
\]

We shall prove that our theorem is also true for \( G(P) \).