A STRUCTURE THEOREM FOR WEAK BUILDINGS OF SPHERICAL TYPE

Abstract. Following earlier work of Tits [8], this paper deals with the structure of buildings which are not necessarily thick; that is, possessing panels (faces of codimension 1) which are contained in two chambers, only. To every building $\Delta$, there is canonically associated a thick building $\tilde{\Delta}$ whose Weyl group $W(\tilde{\Delta})$ can be considered as a reflection subgroup of the Weyl group $W(\Delta)$ of $\Delta$. One can reconstruct $\Delta$ from $\tilde{\Delta}$ together with the embedding $W(\tilde{\Delta}) \subset W(\Delta)$. Conversely, if $\tilde{\Delta}$ is any thick building and $W$ any reflection group containing $W(\Delta)$ as a reflection subgroup, there exists a weak building $\Delta$ with Weyl group $W$ and associated thick building $\tilde{\Delta}$.

In [8], it is proved that there exists no thick building with diagram $H_3$ or $H_4$. A short section of that paper is devoted to the classification of all weak buildings with these two diagrams. The case of weak buildings of rank 2 had already been treated in [7, §1.2]. The classification is reduced to the classification of buildings with a smaller Weyl group. For instance in the case of $H_3$, one reduces to the classification of generalized $n$-gons, $n = 3, 5$, and to the 'classification' of buildings of type $A_1 \times A_1 \times A_1$. (This last case is missing in [8].) The question of classifying weak buildings (in this paper, a building is not necessarily thick, and a non thick building is called weak) of course also arises for all other spherical diagrams, where thick buildings do exist. It is easily suggested that a similar result as for $H_3$ or $H_4$ should hold for any other given diagram, and that one should formulate also a general kind of result which is independent of the particular diagram. I am indebted to J. Tits for a short but fruitful discussion on this problem several years ago.

Recently, the subject of weak buildings has been brought to my attention, again, by S. Rees [4]. She has announced an approach to the classification of weak buildings which is independent of [8] and different in spirit. The results of the present note have been obtained independently of her announcement. I go back to the ideas that presumably have been used by Tits to settle the special cases of $H_3$ and $H_4$, and arrive at a general structure theorem that seems to be a useful extension of S. Rees' result. My result is best understood in the context of geometrical realizations, where the simplices of the building are not regarded as flags of an abstract geometry, but as 'geometrical' simplices, all simplices of one apartment forming a topological sphere. My result says that to each weak building $\Delta$ of rank $n$ there is canonically associated a thick building $\tilde{\Delta}$ of rank $\tilde{n} \leq n$.

and $\Delta$ is obtained from $\Delta$ by forming the join of $\Delta$ with a sphere of dimension $n - \bar{n} - 1$ and then subdividing all the chambers of $\Delta \ast S^{n-\bar{n}-1}$ in the same way. In the case $\bar{n} = n - 1$, this join is the 'suspension' considered by Tits in [8]. The subdivision of a fixed chamber is described by an inclusion $W(\Delta) \to W(\Delta)$ of Weyl groups. Conversely, each thick building $\Delta$ and each inclusion $W(\Delta) \subseteq W$ of reflection groups (in the sense that all the reflections in $W(\Delta)$ are reflections in $W$) gives rise to a weak building with Weyl group $W$ and associated thick building $\Delta$. We wish to emphasize the fact that the case $n = \bar{n}$ in general occurs.

Indeed, it is a standard fact [6, Th. 7.12(ii)] that a building of type $D_n$ gives rise to a weak building of type $C_n$, and it should be well known (e.g. from the theory of regular polytopes) that a building of type $C_4$ gives rise to a weak building of type $F_4$ by considering the shadow geometry

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\text{(cf. [6, 10.14]; [5, §5]). Also, a building of type $C_n \times C_m$ gives rise to a weak building of type $C_{n+m}$ by considering the shadow geometry}
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This construction is equivalent to the concept of a 'direct sum' of two polar spaces as given in [2, p. 226] (cf. [5, §5, Ex. 3]).

We shall see below that the reflection group of type $H_4$ contains subgroups of types $A_4$, $D_4$, $A_2 \times A_2$, $A_1 \times A_1 \times A_1 \times A_1$ all of which admit thick buildings. Therefore, there exist lots of buildings of type $H_4$ with $\bar{n} = n = 4$. They cannot be considered as subdivisions of suspensions of rank-3 buildings. This fact contradicts the claim about $H_4$ in [8].

In the following, a good knowledge of [6, Chs 2, 3] is presupposed. We only recall a few basic facts and introduce some special notions. Let $\Delta$ denote a building of spherical type; that is, a numbered complex with a system of 'apartments' which are finite Coxeter complexes and subject to the conditions (B2), (B3), (B4) of [6, Ch. 3]. The type set is denoted by $I$. The canonical Weyl group of $\Delta$ is the Coxeter group $W(\Delta) = \langle i \mid i \in I, (ij)^{m_{ij}} = 1 \rangle$, where $(m_{ij})_{i,j \in I}$ is the diagram of $\Delta$. For each apartment $\Sigma$ and each chamber $C \in \Sigma$, there is a unique isomorphism from $W(\Delta)$ onto the Weyl group (full automorphism group) $W(\Sigma)$ of $\Sigma$ which maps $i$ onto the reflection mapping $C$ to its $i$-neighbour.