A CHARACTERIZATION OF J₁ IN TERMS OF ITS GEOMETRY

INTRODUCTION

In [1], Buekenhout defines the notion of an incidence structure belonging to a diagram and gives examples of many of the known simple groups which act as automorphism groups of such incidence structures. Among these is the group $G = J₁$ which acts on an incidence structure $S$ belonging to the diagram $\Omega_3$. Briefly what this means is that $S$ consists of three types of varieties which we can call points, edges and pentagons (represented by the first, second and third nodes of the diagram, respectively), as well as an incidence relation on them satisfying certain axioms.

The existence of this incidence structure $S$ can be seen as follows. It is well-known (see for example D. G. Higman [3], p. 37) that $G$ has a primitive, rank 5 permutation representation on a set $\Omega$ of degree 266, with point stabilizer $\text{PSL}(2, 11)$. The graph $\mathcal{H}$ on $\Omega$ defined with respect to the suborbit of length 11 is undirected and connected and satisfies the following properties: (a) the valency of $\mathcal{H}$ is 11; (b) for any $x \in \Omega$, $G_{\Delta(x)} \cong \text{PSL}(2, 11)$; (c) for some path $(x, y, z)$ of length 2, $x, y, z \in \Omega (x \neq z)$, $G_{xyz}$ fixes a pentagon (circuit of length 5) containing $(x, y, z)$; and (d) the vertices and edges fixed by an involution of $G_{xyz}$ constitute a connected subgraph of $\mathcal{H}$. (Here $\Delta(x)$ denotes the vertices of $\mathcal{H}$ adjacent to $x$.) Now the varieties in $S$ are the vertices and edges of $\mathcal{H}$ as well as the set $\mathcal{P}$ of all pentagons which are images in $G$ of the pentagon of (c) above, and the incidence relation between varieties in $S$ is the obvious incidence induced by the graphical structure of $\mathcal{H}$.

The aim of this paper is the characterization of $J₁$ in terms of its action on a graph $\mathcal{H}$ satisfying the above properties. More precisely we shall prove the following:

**THEOREM 1.** Let $\mathcal{H}$ be a connected, regular graph (finite, undirected, no loops or multiple edges) of girth 5 on a set $\Omega$, and $G \leq \text{Aut}(\mathcal{H})$ such that $G$ and $\mathcal{H}$ satisfy properties (a), (b), (c) and (d) above.

Then $G \cong J₁$.

A remark about assumption (d). If we let $B$ be the stabilizer in $G$ of a maximal flag of $S$ and $N$ the stabilizer in $G$ of an apartment of $S$ which contains this maximal flag, then $B$ and $N$ form a type of BN-pair for this group ([2]) and statement (d) asserts that the varieties in $S$ fixed by $B \cap N$ are
connected under the incidence relation of $S$. This is true also for most of the other examples of groups and incidence structures in Buekenhout’s paper.

Theorem 1 is proved using Theorem 2 below, which is Janko’s original characterization of his group.

**Theorem 2.** (Z. Janko [4]) Let $G$ be a finite group with the following properties:

(i) Sylow 2-subgroups of $G$ are abelian;
(ii) $G$ has no subgroup of index 2; and
(iii) $G$ contains an involution $t$ such that $C_G(t) = \langle t \rangle \times F$, where $F \cong A_5$.

Then $G$ is a simple group isomorphic with $J_1$.

In Section 2 we give the definitions of the terms and notation to be used, while in Section 3 two important graphs and their automorphism groups are discussed. Finally in Section 4 we prove Theorem 1.

It should be remarked that the aforementioned graph on which $J_1$ acts gives an example of a polygonal graph (in fact a 5-gon-graph of valency 11) as defined in [5].

### 2. Definitions and notation

All groups and graphs will be finite and all graphs will be undirected, with no loops or multiple edges. If $\mathcal{H}$ is such a graph with vertex set $\Omega$, then for $x, y \in \Omega$, we write $x \sim y$ to mean $x$ is adjacent to $y$. As mentioned before, $\Delta(x)$ denotes the set of vertices of $\Omega$ adjacent to $x \in \Omega$. If $\Gamma$ is a subset of $\Omega$, the induced subgraph of $\mathcal{H}$ on $\Gamma$ is the maximal subgraph of $\mathcal{H}$ with vertices the set $\Gamma$. The valency of $x$ is $|\Delta(x)|$ and $\mathcal{H}$ is called regular if the valency of each vertex is the same. A path of length $n$ is a sequence $(x_0, x_1, \ldots, x_n)$ of $n + 1$ vertices in $\Omega$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, n - 1$, and $x_i \neq x_{i+2}$, $i = 0, \ldots, n - 2$. This path is called a circuit if it is a closed path (i.e. $x_0 = x_n$ and $x_1 \neq x_{n-1}$), and the girth of $\mathcal{H}$ is the length of the smallest circuit of $\mathcal{H}$. A pentagon is a circuit consisting of 5 distinct vertices. The distance from $x$ to $y$ is the length of the shortest path from $x$ to $y$ (if one exists). We say $\mathcal{H}$ is connected if there is a path from $x$ to $y$ for all $x, y \in \Omega$. By a 2-claw $(x, y, z)$ we mean a path $(y, x, z)$ of length 2, so that $y \sim z$ and $y, z \in \Delta(x)$. The automorphism group of $\mathcal{H}$ will be denoted by $\text{Aut}(\mathcal{H})$.

If $G$ is a group acting on a set $\Omega$, we denote by $x^g$ the image of $x \in \Omega$ by the element $g \in G$. For $W = \{x, y, z, \ldots\}$ a subset of $\Omega$, $G_{xyz\ldots} = G_{[W]}$ will denote the pointwise stabilizer, and $G_W$ the setwise stabilizer, of $W$. $G^\Omega$ will denote the group of permutations of $\Omega$ induced by the action of $G$, so that $G^\Omega \cong G / G_{(\Omega)}$. For $g \in G$, $\Omega(g)$ is the subset of $\Omega$ fixed (pointwise) by $g$. 