ON SURFACE AREA MEASURES OF CONVEX BODIES

ABSTRACT. The set $\mathcal{S}_j$ of $j$th-order surface area measures of convex bodies in $d$-space is well known for $j = d - 1$. A characterization of $\mathcal{S}_1$ was obtained by Firey and Berg. The determination of $\mathcal{S}_j$, for $j \in \{2, \ldots, d - 2\}$, is an open problem. Here we show some properties of $\mathcal{S}_j$ concerning convexity, closeness, and size. Especially we prove that the difference set $\mathcal{S}_j - \mathcal{S}_1$ is dense (in the weak topology) in the set of signed Borel measures on the unit sphere which have barycentre $0$.

0. INTRODUCTION

In the following we are concerned with the set $\mathcal{S}_j$ of $j$th-order surface area measures of convex bodies in Euclidean $d$-space, $j \in \{1, \ldots, d - 1\}$, $d \geq 2$. These measures were introduced in 1937/38 by Aleksandrov [1] and Fenchel and Jessen [5] (see also Busemann [4]) and can be defined as follows: For a convex body $K$ and a Borel set $\omega$ of the unit sphere $\Omega$, the value of the $(d - 1)$th-order surface area measure $\mu_{d-1}(K; \omega)$ is the area of that part of the boundary of $K$ touched by support planes whose outer normal directions fall in $\omega$. In particular, for the vector sum $K + \alpha B$, where $\alpha \geq 0$ and $B$ is the unit ball, $\mu_{d-1}(K + \alpha B; \cdot)$ is a polynomial of degree $d - 1$ in $\alpha$. To within a binomial coefficient, the coefficient of $\alpha^{d-j-1}$ is called the $j$th-order surface area measure $\mu_j(K; \cdot)$ of $K$. The set $\mathcal{S}_j$ is a subset of $\mathcal{M}$, the set of finite Borel measures on $\Omega$ which have barycentre $0$.

Aleksandrov [1] and Fenchel and Jessen [5] showed that precisely the measures in $\mathcal{M}$ which have dimension $d$ or $1$ belong to $\mathcal{S}_{d-1}$ and extended, thus, a theorem of Minkowski on polytopes. About thirty years later Firey [6, 7] and Berg [2] gave a characterization of $\mathcal{S}_1$. Here, the polytopal case was treated in a direct way by Schneider [12]. To date not very much is known about $\mathcal{S}_j$ for $j \in \{2, \ldots, d - 2\}$. In [8], Firey solved the characterization problem for smooth bodies of revolution. In [9] he investigated the local behaviour of the elements of $\mathcal{S}_j$. Schneider [11] determined the support of $j$th-order surface area measures. In the field of integral geometry, Firey and Schneider obtained further results on surface area measures (see [11] for references). A survey of open questions on surface area measures can be found in [10].

Our aim is to give information about the size of the subset $\mathcal{S}_j$ of $\mathcal{M}$. For this purpose, we start in Section 1 with some elementary considerations on the dimensions of surface area measures and convex bodies. Weak limits of surface area measures are studied in Section 2. In Section 3 we first mention that $\mathcal{S}_{d-1}$ is dense in $\mathcal{M}$ (in the weak topology) and $\mathcal{S}_{d-1} - \mathcal{S}_{d-1} = \mathcal{M} - \mathcal{M}$, whereas the corresponding statements are false for $\mathcal{S}_j$, $j < d - 1$. The main
result will be that $\mathcal{S}_j - \mathcal{S}_j$ is dense in $\mathcal{M} - \mathcal{M}$. Hence, the set $\mathcal{S}_j$ is not too small.

Some applications to centrally symmetric convex bodies, zonoids, and generalized zonoids appear in a forthcoming paper [13].

**NOTATIONS**

$E^d$ is the $d$-dimensional Euclidean space with unit ball $B$ and unit sphere $\Omega$.

Let $\mathcal{K}$ be the set of convex bodies in $E^d$ supplied with the Hausdorff metric. For $K \in \mathcal{K}$ let $h_K$ be the support function of $K$. We shall assume throughout that the convex body $K$ contains the origin; thus, $h_K \geq 0$. $V(K_1, \ldots, K_d)$ is the mixed volume of $K_1, \ldots, K_d \in \mathcal{K}$. For $\dim K = j$ let $V_j(K)$ be the $j$-dimensional volume of $K \in \mathcal{K}$. $\mu_j(K; \cdot)$ for $j \in \{1, \ldots, d - 1\}$ is the $j$th-order surface area measure of $K$. Let $\mathcal{S}_j$ be the set $\{\mu_j(K; \cdot) \mid K \in \mathcal{K}\}$.

By $\mathcal{M}$ we denote the set of finite Borel measures on $\Omega$ with barycentre 0. $\mathcal{M}$ is supplied with the weak topology; $\supp \mu$ is the support of the measure $\mu$.

$\dim K$ (resp. $\dim \mu$) is the dimension of the linear hull of $K \in \mathcal{K}$ (resp. $\supp \mu$). $\perp K$ (resp. $\perp \supp \mu$) means orthogonal to the linear hull of $K \in \mathcal{K}$ (resp. $\supp \mu$).

For $u \in E^d$ let $\bar{u}$ be the segment $[0, u]$. ‘cl’ denotes the closure of a set; ‘conv’ denotes the convex hull.

### 1. THE CONNECTION BETWEEN $\dim K$ AND $\dim \mu_j(K; \cdot)$

The following considerations are based on the fundamental equation which relates surface area measures and mixed volumes of convex bodies (see [1] and [5]):

(1.1) $V(L, K_1, \ldots, K_j, B, \ldots, B) = \frac{1}{d} \int_{\Omega} h_l(x)\mu_j(K_j; dx)$ for all $K, L \in \mathcal{K}$ and all $j \in \{1, \ldots, d - 1\}$.

From well-known results about mixed volumes (Bonnesen and Fenchel [3, p. 41]) we deduce:

(1.2) **PROPOSITION.** (a) $\dim K \geq j + 1$ if and only if $\dim \mu_j(K; \cdot) = d$.

(b) $\dim K = j$ if and only if $\dim \mu_j(K; \cdot) = d - j$.

(c) $\dim K \leq j - 1$ if and only if $\dim \mu_j(K; \cdot) = 0$.

In case (b) we have, moreover,

\[(*) \quad \mu_j(K; \cdot) = \frac{d}{\binom{d}{j}} V_j(K) \cdot \lambda_{d-j},\]

where $\lambda_{d-j}$ is the $(d - j)$-dimensional Lebesgue measure concentrated on the sphere orthogonal to $K$. 