ON SURFACE AREA MEASURES OF CONVEX BODIES

ABSTRACT. The set $\mathcal{S}_j$ of $j$th-order surface area measures of convex bodies in $d$-space is well known for $j = d - 1$. A characterization of $\mathcal{S}_1$ was obtained by Firey and Berg. The determination of $\mathcal{S}_j$, for $j \in \{2, \ldots, d-2\}$, is an open problem. Here we show some properties of $\mathcal{S}_j$ concerning convexity, closeness, and size. Especially we prove that the difference set $\mathcal{S}_j - \mathcal{S}_j$ is dense (in the weak topology) in the set of signed Borel measures on the unit sphere which have barycentre 0.

0. INTRODUCTION

In the following we are concerned with the set $\mathcal{S}_j$ of $j$th-order surface area measures of convex bodies in Euclidean $d$-space, $j \in \{1, \ldots, d - 1\}$, $d \geq 2$. These measures were introduced in 1937/38 by Aleksandrov [1] and Fenchel and Jessen [5] (see also Busemann [4]) and can be defined as follows: For a convex body $K$ and a Borel set $\omega$ of the unit sphere $\Omega$, the value of the $(d - 1)$th-order surface area measure $\mu_{d-1}(K; \omega)$ is the area of that part of the boundary of $K$ touched by support planes whose outer normal directions fall in $\omega$. In particular, for the vector sum $K + \alpha B$, where $\alpha \geq 0$ and $B$ is the unit ball, $\mu_{d-1}(K + \alpha B; \cdot)$ is a polynomial of degree $d - 1$ in $\alpha$. To within a binomial coefficient, the coefficient of $\alpha^{d-j-1}$ is called the $j$th-order surface area measure $\mu_j(K; \cdot)$ of $K$. The set $\mathcal{S}_j$ is a subset of $\mathcal{M}$, the set of finite Borel measures on $\Omega$ which have barycentre 0.

Aleksandrov [1] and Fenchel and Jessen [5] showed that precisely the measures in $\mathcal{M}$ which have dimension $d$ or 1 belong to $\mathcal{S}_{d-1}$ and extended, thus, a theorem of Minkowski on polytopes. About thirty years later Firey [6, 7] and Berg [2] gave a characterization of $\mathcal{S}_1$. Here, the polytopal case was treated in a direct way by Schneider [12]. To date not very much is known about $\mathcal{S}_j$ for $j \in \{2, \ldots, d - 2\}$. In [8], Firey solved the characterization problem for smooth bodies of revolution. In [9] he investigated the local behaviour of the elements of $\mathcal{S}_j$. Schneider [11] determined the support of $j$th-order surface area measures. In the field of integral geometry, Firey and Schneider obtained further results on surface area measures (see [11] for references). A survey of open questions on surface area measures can be found in [10].

Our aim is to give information about the size of the subset $\mathcal{S}_j$ of $\mathcal{M}$. For this purpose, we start in Section 1 with some elementary considerations on the dimensions of surface area measures and convex bodies. Weak limits of surface area measures are studied in Section 2. In Section 3 we first mention that $\mathcal{S}_{d-1}$ is dense in $\mathcal{M}$ (in the weak topology) and $\mathcal{S}_{d-1} - \mathcal{S}_{d-1} = \mathcal{M} - \mathcal{M}$, whereas the corresponding statements are false for $\mathcal{S}_j, j < d - 1$. The main
result will be that $\mathcal{I}_j - \mathcal{I}_j$ is dense in $\mathcal{M} - \mathcal{M}$. Hence, the set $\mathcal{I}_j$ is not too small.

Some applications to centrally symmetric convex bodies, zonoids, and generalized zonoids appear in a forthcoming paper [13].

**NOTATIONS**

$E^d$ is the $d$-dimensional Euclidean space with unit ball $B$ and unit sphere $\Omega$.

Let $\mathcal{K}$ be the set of convex bodies in $E^d$ supplied with the Hausdorff metric. For $K \in \mathcal{K}$ let $h_K$ be the support function of $K$. We shall assume throughout that the convex body $K$ contains the origin; thus, $h_K \geq 0$. $V(K_1, \ldots, K_d)$ is the mixed volume of $K_1, \ldots, K_d \in \mathcal{K}$. For $\dim K = j$ let $V_j(K)$ be the $j$-dimensional volume of $K \in \mathcal{K}$. $\mu_j(K; \cdot)$ for $j \in \{1, \ldots, d-1\}$ is the $j$th-order surface area measure of $K$. Let $\mathcal{I}_j$ be the set $\{\mu_j(K; \cdot) \mid K \in \mathcal{K}\}$.

By $\mathcal{M}$ we denote the set of finite Borel measures on $\Omega$ with barycentre 0. $\mathcal{M}$ is supplied with the weak topology; supp $\mu$ is the support of the measure $\mu$.

$\dim K$ (resp. $\dim \mu$) is the dimension of the linear hull of $K \in \mathcal{K}$ (resp. supp $\mu$). $\perp K$ (resp. $\perp$ supp $\mu$) means orthogonal to the linear hull of $K \in \mathcal{K}$ (resp. supp $\mu$).

For $u \in E^d$ let $\bar{u}$ be the segment $[0, u]$. ‘$\overline{\cdot}$’ denotes the closure of a set; ‘conv’ denotes the convex hull.

1. **THE CONNECTION BETWEEN $\dim K$ AND $\dim \mu_j(K; \cdot)$**

The following considerations are based on the fundamental equation which relates surface area measures and mixed volumes of convex bodies (see [1] and [5]):

\[
(1.1) \quad V(L, K_1, \ldots, K_d, B, \ldots, B) = \frac{1}{d} \int_{\Omega} h_L(x) \mu_j(K; dx) \text{ for all } K, L \in \mathcal{K} \text{ and all } j \in \{1, \ldots, d-1\}.
\]

From well-known results about mixed volumes (Bonnesen and Fenchel [3, p. 41]) we deduce:

\[
(1.2) \quad \text{PROPOSITION. (a) } \dim K \geq j + 1 \text{ if and only if } \dim \mu_j(K; \cdot) = d.
\]

(b) \quad $\dim K = j$ if and only if $\dim \mu_j(K; \cdot) = d - j$.

(c) \quad $\dim K \leq j - 1$ if and only if $\dim \mu_j(K; \cdot) = 0$.

In case (b) we have, moreover,

\[
(*) \quad \mu_j(K; \cdot) = d \left(\frac{d}{j}\right) \cdot V_j(K) \cdot \lambda_{d-j},
\]

where $\lambda_{d-j}$ is the $(d-j)$-dimensional Lebesgue measure concentrated on the sphere orthogonal to $K$. 