CAMERON–LIEBLER LINE CLASSES
IN PG(3, q)

ABSTRACT. A Cameron–Liebler line class is a set $L$ of lines in PG(3, q) for which there exists a
number $x$ such that $|L \cap S| = x$ for all spreads $S$. There are many equivalent properties: Theorem
1 lists eight. This paper classifies Cameron–Liebler line classes with $x \leq 4$ (with two exceptions).
It is also shown that the study of Cameron–Liebler line classes is equivalent to the study of
weighted sets of points in PG(3, q) with two weights on lines.

Throughout this paper, $\mathcal{P}$ denotes the set of points, $\mathcal{L}$ the set of lines and $\Pi$ the
set of planes of PG(3, q). The incidence map of PG(3, q) is the linear map
$$\alpha: \mathbb{Q}^\mathcal{P} \to \mathbb{Q}^\mathcal{L}$$
with $\ell(\alpha x) = \sum_{P \in \ell} Pf$ for all $\ell \in \mathcal{L}$, $f \in \mathbb{Q}^\mathcal{P}$. The dual incidence map of PG(3, q)
is the linear map
$$\alpha^*: \mathbb{Q}^\mathcal{L} \to \mathbb{Q}^\mathcal{P}$$
with $P(\alpha^* x) = \sum_{P \in \ell} \ell f$ for all $P \in \mathcal{P}$, $f \in \mathbb{Q}^\mathcal{L}$. With respect to the usual bases of
$\mathbb{Q}^\mathcal{P}$ and $\mathbb{Q}^\mathcal{L}$, $\alpha$ is represented by an incidence matrix of the point–line
structure of PG(3, q), and $\alpha^*$ is represented by the transpose of one.

Note that $\mathbb{Q}^\mathcal{P} = \text{im} \alpha \oplus \ker \alpha^*$, and that this sum is orthogonal with
respect to the usual form
$$(f, g) = \sum_{\ell \in \mathcal{L}} \ell f \ell g.$$ 
Throughout $e$ denotes the constant function with value 1. Sometimes $e$ will
have domain $\mathcal{P}$; at other times it will have domain $\mathcal{L}$. This should be clear
from the context.

Since $G = \text{PGL}(4, q)$ preserves incidence, $\langle e \rangle$, $\langle e \rangle^\perp \cap \text{im} \alpha$ and $\ker \alpha^*$ are
all modules over $\mathbb{Q}G$. Since $G$ has rank 3 on lines, they are all irreducible.

Finally, for $P \in \mathcal{P}$, $\text{star}(P) = \{ \ell \in \mathcal{L}: P \in \ell \}$; for $\pi \in \Pi$, $\pi_\mathcal{L} = \{ \ell \in \mathcal{L}: \ell \subseteq \pi \}$;
and if $P \in \pi$, pencil$(P, \pi) = \{ \ell \in \mathcal{L}: P \in \ell \subseteq \pi \}$.

The first result gives many equivalent conditions on a set of lines in
PG(3, q). Parts (i) to (v) were proved equivalent in [2].

THEOREM 1. Let $L$ be a set of lines of PG(3, q). The following are equivalent:

(i) $\chi_L \in \text{im} \alpha$.
(ii) $\chi_L \in (\ker \alpha^*)^\perp$.

There exists $x$ such that $|L \cap S| = x$ for all spreads $S$.

There exists $x$ such that $|L \cap S| = x$ for all regular spreads $S$.

$|L \cap R| = |L \cap R^{opp}|$ for all reguli $R$.

There exists $x$ such that
$$|\text{star}(P) \cap L| + |\pi_{\not\subseteq} \cap L| = x + (q+1)|\text{pencil}(P, \pi) \cap L|.$$ 

There exists $x$ such that
$$|\{m \in L : m \text{ meets } \ell, m \neq \ell\}| = (q+1)x + (q^2-1)x_L,$$
for all $\ell \in L$.

There exists $x$ such that
$$|\{n \in L : n \text{ is a transversal to } \ell \text{ and } m\}| = x + q(\ell x_L + m x_L),$$
for all $\ell, m \in L$, with $\ell$ and $m$ skew.

**Proof.** Since $\text{im } \alpha = (\ker \alpha^*)^\perp$, the equivalence of (i) and (ii) follows. For the other parts, we apply the fact that $\ker \alpha^*$ is an irreducible $\text{PGL}(4, q)$-module. It follows that $f \in (\ker \alpha^*)^\perp$ if, and only if, $f \perp g_{\text{PGL}(4, q)}$, for some $g \in \ker \alpha^*$. We need only display the appropriate functions $g$ for the various parts, and derive the equations from the fact that $x_L \perp g_{\text{PGL}(4, q)}$.

For (iii) and (iv), take $g$ to be $x_s - e/(q^2 + q + 1)$. This is in $\ker \alpha^*$, as $P x s = \sum_{P \in \ell} x_s/k s = 1$. Since $\text{PGL}(4, q)$ takes spreads to spreads, and regular spreads to regular spreads, $x_L$ is perpendicular to all such $g$ if, and only if, $x_L \in (\ker \alpha^*)^\perp$. But $(x_L, g) = |L \cap S| - |L|/(q^2 + q + 1)$. So (ii) and (iii) are equivalent, as are (ii) and (iv).

For (v), take $g$ to be $x_R - x_{\text{R}^{opp}}$.

For (vi), take $g$ to be
$$x_{\text{star}(P)} + x_{\pi_{\not\subseteq}} - (q+1)x_{\text{pencil}(P, \pi)} - e/(q^2 + q + 1).$$

For (vii), take $g$ to be
$$x_M - (q^2 - 1)x_\ell - (q+1)e/(q^2 + q + 1),$$
where $M = \{m \in \ell : m \text{ meets } \ell, m \neq \ell\}$.

For (viii), take $g$ to be
$$x_N = q x_{(\ell, m)} - e/(q^2 + q + 1),$$
where $N = \{n \in \mathcal{L} : n \text{ is a transversal to } \ell \text{ and } m\}$.

We call a set $L$ of lines satisfying conditions (i) to (viii) a **Cameron–Liebler line class** [2]. (These were called special line classes there.) It follows from the details of the proof that it is the same $x$ in each of the parts, and that $|L| = (q^2 + q + 1)x$. We call $x$ the **parameter** of $L$. Note that the complement