A CHARACTERIZATION OF THE EUCLIDEAN BALL IN TERMS OF CONCURRENT SECTIONS OF CONSTANT WIDTH

ABSTRACT. We prove that the Euclidean ball is the unique convex body with the property that all its sections through a fixed point are convex bodies of constant width. Furthermore, we characterize those convex bodies which are sections of convex bodies of constant width.

1. INTRODUCTION AND NOTATION

The main purpose of this paper is to prove the following theorem:

THEOREM 1. Let $2 \leq k < n$, let $K \subset \mathbb{E}^n$ be an $n$-dimensional convex body and let $p_0$ be a point of $\mathbb{E}^n$ with the property that every $k$-section of $K$ through $p_0$ is a convex body of constant width. Then $K$ is a Euclidean $n$-ball.

When we say that every $k$-section of $K$ through $p_0$ is a convex body of constant width we mean that if $\Pi$ is a $k$-plane of $\mathbb{E}^n$ through $p_0$, then $\Pi \cap K$ is either empty, a single point or a $k$-dimensional convex body of constant width.

Under differentiability conditions on $K$ and its sections, Theorem 1 was proved by Süss [2], when $n = 3$ and $p_0 \in \text{int } K$.

A $k$-dimensional convex body $K \subset \mathbb{E}^k$ will be called $R$-convex, if there is a sufficiently large number $r > 0$ and for every $p \in \text{bdr } K$ there is a Euclidean $k$-ball $B \subset \mathbb{E}^k$ of diameter $r$ which contains $K$ and has $p$ on its boundary. By Theorem 1 we know that not every $k$-section of a convex body of constant width, different from a Euclidean ball, has constant width, thus it would be interesting to know which convex bodies are $k$-sections of convex bodies of constant width. In this direction we shall prove that a $k$-dimensional convex body $K$ is the $k$-section of an $n$-dimensional convex body of constant width if and only if $K$ is $R$-convex.

In this paper $\mathbb{E}^n$ will denote Euclidean $n$-space, where it is always assumed that $n \geq 2$. A Euclidean $n$-ball in $\mathbb{E}^n$, or simply a Euclidean ball in $\mathbb{E}^n$, will be a subset of $\mathbb{E}^n$ homothetic to $\{x \in \mathbb{E}^n \mid \|x\| \leq 1\}$. If $a, b \in \mathbb{E}^n$, then $[a, b]$ and $(a, b)$ will denote the closed and open intervals with extreme points $a$ and $b$, respectively. If $K$ is a $(k + 1)$-dimensional convex body and $[a, b]$ is a chord of

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K, then \([a, b]\) will be called a \textit{binormal} of \(K\) provided that the \(k\)-planes through \(a\) and \(b\), orthogonal to \([a, b]\), are support \(k\)-planes of \(K\). Furthermore, \([a, b]\) is a \textit{diameter} of \(K\) if \([a, b]\) is a chord of maximal length. Of course, for convex bodies of constant width the concepts of binormal and diameter coincide. For further notation see [1].

The following is an outline of the proof of Theorem 1. First we shall prove (Lemma 3) that \(K\) has constant width. The idea is to prove that all binormals of \(K\) are concurrent. For that purpose, we shall prove (Lemma 4) that there is a fixed binormal \(B_0\) of \(K\) with the property that any other binormal \(B\) of \(K\) intersects \(B_0\) in some point. Next, for every point \(x_0 \in B_0\), we shall study the structure of the set of all binormals of \(K\) which contain \(x_0\). If this set is not empty, its topological structure will be so rich (Lemma 7) that connectivity arguments on the boundary of \(K\) will allow us to conclude (Section 3) that every 2-plane through \(B_0\) intersects \(K\) in a Euclidean 2-ball. Our arguments, in this last part of the proof, are different depending on whether \(p_0 \in \text{int} \ K\), \(p_0 \in \text{bd} \ K\) or \(p_0 \notin K\).

Hence, Section 2 will be devoted to the study of the intersection structure of the binormals of \(K\) and in Section 3 we shall use these results to prove Theorem 1. Finally, in Section 4 we shall characterize those convex bodies which are sections of convex bodies of constant width.

If \(2 \leq k < n\) and \(K\) is an \(n\)-dimensional convex body with the property that all \(k\)-sections through a point \(p_0 \in \mathbb{E}^n\) are Euclidean balls, then it is known that \(K\) must be a Euclidean ball. Consequently, for the proof of Theorem 1 it will be enough to consider only the case when \(K\) is a \((k + 1)\)-dimensional convex body.

2. Preliminary lemmas

From now on \(K\) is a \((k + 1)\)-dimensional convex body, \(k \geq 2\), and \(p_0\) is a point of \(\mathbb{E}^{k+1}\) with the property that every \(k\)-section of \(K\) through \(p_0\) has constant width.

LEMMA 1. \(\text{There is a line } L_0 \text{ through } p_0 \text{ such that } B_0 = L_0 \cap K \text{ is a diameter of } K.\)

\textit{Proof.} Let \(L\) be a line with the property that \(B = L \cap K\) is a diameter of \(K\). If \(p_0 \in L\) there is nothing to prove. If \(p_0 \notin L\), let \(H\) be a \(k\)-plane through \(p_0\) and \(L\). By hypothesis \(H \cap K\) has constant width. Then, there is a line \(L_0\) in \(H\) through \(p_0\) with the property that \(B_0 = L_0 \cap (H \cap K)\) is a binormal and hence a diameter of \(H \cap K\). Since \(B\) is also a binormal or diameter of \(H \cap K\), \(B\) and \(B_0\) must have the same length and therefore \(B_0 = L_0 \cap K\) is a diameter of \(K\). \(\square\)