ON THE HOMOGENEOUS RIEMANNIAN
STRUCTURES OF TYPE $\mathcal{F}_1 \oplus \mathcal{F}_3$

ABSTRACT. We study homogeneous Riemannian structures belonging to the class $\mathcal{F}_1 \oplus \mathcal{F}_3$ of the classification given by Tricerri and Vanhecke. The main result is the following: a connected, simply connected Riemannian manifold $M$ admits a homogeneous structure $T$ of type $\mathcal{F}_1 \oplus \mathcal{F}_3$, if and only if $M$ is isometric to a hyperbolic space $H^n$.

INTRODUCTION

The notion of a homogeneous Riemannian structure on a Riemannian manifold $(M, g)$ has been introduced in [6]. The manifold $M$ is supposed to be connected, of class $C^\infty$, and of dimension $n \geq 2$. Such a homogeneous structure is a tensor field $T$ of type $(1, 2)$ on $M$, which satisfies the following equations of Ambrose and Singer:

\begin{align*}
(A-S) & \quad (i) \ g(T_X Y, Z) + g(T_X Z, Y) = 0 \\
& \quad (ii) \ (V_X R)(Y) = [T_X, R_Y] - R_{T_X Y} - R_{Y T_X} \\
& \quad (iii) \ (V_X T)(Y) = [T_X, T_Y] - T_{T_X Y}
\end{align*}

for all $X, Y, Z \in \mathfrak{X}(M)$. Here, $\mathfrak{X}(M)$ is the Lie algebra of the tangent vector fields on $M$, $V$ denotes the Riemannian connection and $R$ is the Riemann curvature tensor, defined by

$$R(X, Y) = [V_X, V_Y] - V_{[X, Y]}.$$

It is well known that, putting $\nabla = V - T$, an equivalent formulation of (A-S) is given by:

\begin{align*}
(A-S) & \quad (i) \ \nabla g = 0 \\
& \quad (ii) \ \nabla R = 0 \\
& \quad (iii) \ \nabla T = 0.
\end{align*}

Using representation theory, F. Tricerri and L. Vanhecke determined a classification of the homogeneous structures in eight classes. For further details, see [6]. Among these eight classes, the class $\mathcal{F}_1 \oplus \mathcal{F}_3$ is characterized by the following condition on $T$:

$$T(X, Y) + T(Y, X) = 2g(X, Y)\xi - g(X, \xi)Y - g(Y, \xi)X$$

with $\xi \in \mathfrak{X}(M)$. Moreover, it is already known that, a complete, connected and
simply connected Riemannian manifold admits a homogeneous structure $T$ of type $\mathcal{F}_3$ if and only if it is a naturally reductive homogeneous Riemannian manifold (see [6, Ch. 6]).

This paper is concerned with the study of the homogeneous structures of type $\mathcal{F}_1 \oplus \mathcal{F}_3$, which do not belong to $\mathcal{F}_3$. The main result will be the following theorem (see Section 2):

**THEOREM.** An $n$-dimensional, connected and simply connected Riemannian manifold admits a homogeneous structure $T \in \mathcal{F}_1 \oplus \mathcal{F}_3$, $T \notin \mathcal{F}_1$, $T \notin \mathcal{F}_3$ if and only if it is isometric to the hyperbolic space $\mathbb{H}^n$ and $n \geq 4$.

Note that the analogous result holds for non-zero structures in the class $\mathcal{F}_1$ with no restriction on the dimension of $M$ (see [6], [8]). More generally, in [7] it is proved that when a homogeneous structure $T$ satisfies the conditions

$$T_X \xi = g(X, \xi)\xi - \|\xi\|^2 X, \quad T_\xi X = 0$$

for all $X \in \mathfrak{X}(M)$, where $\xi \neq 0$ is defined by

$$(n - 1)\xi = \sum_{i=1}^{n} T(E_i, E_i)$$

with $\{E_1, \ldots, E_n\}$ arbitrary orthonormal local fields, then the supporting Riemannian manifold is locally isometric to the hyperbolic space $\mathbb{H}^n$ of constant sectional curvature $k = -\|\xi\|^2$. Using this result, the necessary condition of the above theorem will be established in Section 1. Section 2 contains the construction of homogeneous structures $T \in \mathcal{F}_1 \oplus \mathcal{F}_3$, $T \notin \mathcal{F}_1$, $T \notin \mathcal{F}_3$, on $\mathbb{H}^n$. Finally, in Section 3 it is proved that the vector space of the real functions on $\mathbb{H}^n$, which are affine with respect to $\xi$, has dimension 2 and admits the set $\{1, f\}$, where $\text{grad } f = \xi$, as a base.

1. **Riemannian Manifolds with a Homogeneous Structure in $\mathcal{F}_1 \oplus \mathcal{F}_3$**

Let $(M, g)$ be an $n$-dimensional, complete, connected, Riemannian manifold equipped with a homogeneous structure $T \in \mathcal{F}_1 \oplus \mathcal{F}_3$, $T \notin \mathcal{F}_3$. From the definition of the class $\mathcal{F}_1 \oplus \mathcal{F}_3([6])$, it follows that for each $X, Y \in \mathfrak{X}(M),$

$$T(X, Y) + T(Y, X) = 2g(X, Y)\xi - g(Y, \xi)X - g(X, \xi)Y$$

and hence

$$T(X, Y) = g(X, Y)\xi - g(Y, \xi)X + \pi(X, Y),$$

where $\xi$ is a non-zero vector field and $\pi$ is a skew-symmetric tensor field of type $(1,2)$. 