A CHARACTERIZATION OF CURVES 
of cyclic order four

ABSTRACT. Differentiable curves of cyclic order 4 are well known as regards the number and type of singular points. This paper characterizes curves of cyclic order 4 in the real conformal plane when no differentiability conditions are assumed by introducing multiplicities for singular points. It is shown that the sum of the multiplicities of the singular points is exactly four.

1. Introduction

1.1. It is known that a strongly differentiable curve \( \mathcal{C} \) of cyclic order 4 in the real conformal plane contains exactly four singular points ([1, 4.1.4.3]). Assuming no differentiability conditions, the best bound obtained for the number of singular points was eleven ([4] and [1, 4.1.3]) and then reduced to the best possible bound; namely four ([5, 4.1]).

In [6] multiplicities are introduced for singular points on such arcs and curves with the result that the sum of the multiplicities of all singular points is indeed at most four. In this paper it is shown that the sum of the multiplicities of all singular points on a curve \( \mathcal{C} \) of cyclic order 4 is exactly four (cf. Theorems 3 and 4).

1.2 DEFINITION. A point \( p \) on an arc \( \mathcal{A} \) in the real conformal plane is said to be (conformally) differentiable [2] if it satisfies two conditions:

I. For every point \( R \neq p \) and if \( s \to p \) (s converges to \( p \)) on \( \mathcal{A} \) there exists a circle \( C_0 \) such that \( C(s,p,R) \to C_0 \). \( C_0 \) is called the tangent circle of \( \mathcal{A} \) at \( p \) through \( R \) and is denoted \( C(p^2,R) \).

II. If \( s \to p \) on \( \mathcal{A} \) there exists a circle \( C(p^3) \) such that \( C(p^2,s) \to C(p^3) \). \( C(p^3) \) is called the osculating circle of \( \mathcal{A} \) at \( p \). \( C(p^3) \) may be the point circle \( p \). For simplicity \( C(p^3) \) will be abbreviated to \( C(p) \).

DEFINITION. A point \( p \) on \( \mathcal{A} \) is said to be strongly differentiable if the following are satisfied:

CI'. Let \( R \neq p, Q \to R \). If \( u,v \to p \) on \( \mathcal{A} \), then \( C(u,v,Q) \) converges.

CII'. \( C(t,u,v) \) converges if the three distinct points \( t,u,v \to p \) on \( \mathcal{A} \).

The following known results concerning differentiability can be found in [3]:

(i) All tangent circles of \( \mathcal{A} \) at \( p \) meet only at \( p \).

(ii) Non-tangent circles of \( \mathcal{A} \) at \( p \) either all intersect \( \mathcal{A} \) at \( p \) or all support \( \mathcal{A} \) at \( p \) (a circle \( C \) intersects or supports an arc \( \mathcal{A} \) at \( p \) provided that there exists a two-sided neighbourhood \( L \cup \{p\} \cup M \)
of \( p \) on \( \mathcal{A} \) such that \( L \) and \( M \) lie on opposite sides or the same side of \( C \), respectively.

(iii) The non-osculating tangent circles of \( \mathcal{A} \) at \( p \) either all intersect or all support \( \mathcal{A} \) at \( p \). If \( C(p) \neq p \), all support.

(iv) Strong differentiability implies ordinary differentiability.

(v) If each point \( p \) of \( \mathcal{A} \) is strongly differentiable then the osculating circle \( C(p) \) varies continuously with \( p \).

1.3. DEFINITION. A differentiable interior point \( p \) of an arc \( \mathcal{A} \) has the characteristic \((a_0, a_1, a_2)\) if \( C(p) \neq p \) or \((a_0, a_1, a_2)_0\) if \( C(p) = p \) where:

(i) \( a_0, a_1, a_2 \) are equal to 1 or 2;

(ii) \( a_0[\lfloor a_0 + a_1 \rfloor] \) is even or odd accordingly as the non-tangent circles [the non-osculating tangent circles] of \( \mathcal{A} \) at \( p \) support or intersect \( \mathcal{A} \) at \( p \);

(iii) \( a_0 + a_1 + a_2 \) is even if \( C(p) \) supports \( \mathcal{A} \) at \( p \), odd if \( C(p) \) intersects \( \mathcal{A} \) at \( p \).

1.4. DEFINITION. The cyclic order of an arc \( \mathcal{A} \) is the maximum number of points in common with any circle. The order of a point \( p \) is the minimum of the orders of all neighbourhoods of \( p \) on \( \mathcal{A} \).

A point of (minimal) order 3 is called an ordinary point, a point of order greater than 3 a singular point, and a point of support of \( \mathcal{A} \) with respect to \( C(p) \) a vertex.

This paper involves only (simple) arcs \( \mathcal{A} \) of cyclic order 4 in the real conformal plane. With regard to such arcs it is known that:

(i) there are at most four singular points [5],

(ii) differentiable points with the characteristic \((1,1,1)\) are exactly the ordinary points while differentiable singular points are vertices and have the characteristic \((1,1,2), (1,1,2)_0, (1,2,1)_0, (2,1,1)_0\) [3],

(iii) points with the characteristic \((1,1,1), (1,1,2) \) or \((1,1,2)_0\) are moreover strongly differentiable [3],

(iv) \( \mathcal{A} \) is one-sidedly strongly differentiable at each interior point ([4, 3.4]); i.e. the two one-sided osculating circles (which are identical for a differentiable point) exist.

REMARK. For the balance of the paper \( \mathcal{A} \) will denote an oriented arc of cyclic order 4.

1.5. DEFINITION. Let \( a \) be any point of \( \mathcal{A} \) and \( \Gamma_a \) the system of all circles passing through \( a \). Then a point \( y \) of \( \mathcal{A} \) is said to be \( \Gamma_a\)-singular ([5, 3.1]) if,