ON TILINGS OF THE PLANE

ABSTRACT. The paper discusses homeomorphic types of (periodic) tilings of the plane in terms of their associated Delaney symbol. Such a symbol consists of a (finite) set $\mathcal{D}$ on which three involutions $\sigma_1, \sigma_2,$ and $\sigma_3$ act from the right such that $\sigma_3\sigma_1 = \sigma_2\sigma_3$ and there are two maps $m_0, m_1, m_2 : \mathcal{D} \to \mathbb{N}$ satisfying certain compatibility conditions. It is shown how the barycentric subdivision of a tiling can be used to define its Delaney symbol and that the symbol characterizes the tiling up to (equivariant) homeomorphisms. Furthermore, it is shown how properties of the tiling can be recognized from corresponding properties of the symbol and how this technique can be used to enumerate various types of tilings with specific properties. If necessary, this enumeration can be done by appropriate computer programs. Among other results, we have been able to vindicate the results by Grünbaum et al., announced in [8]. Finally, some recursive enumeration formulas, based on the Delaney symbol technique, are stated.

0. INTRODUCTION

In [8] the authors carefully exhibit 508 different types of '2-homeohedral, normal' tilings of the Euclidean plane $\mathbb{E} = \mathbb{E}^2$ and state: 'It would be nice to be able to assert that the enumeration presented here is complete. Unfortunately we cannot do that, though it seems reasonable to suppose that the number of omissions (if any) must be quite small... Practically speaking, it seems that there is no way of doing this in a completely systematic manner.' We shall see below that in fact such a systematic method does exist, and show how it can be used to supply the missing proofs. More precisely, we want to explain how the theory of Delaney symbols of tilings (cf. [2]–[4], [6], [7]) can be used efficiently to reduce enumeration problems like the one mentioned above to a finite, purely combinatorial problem which can be solved easily by appropriate computer programs. In addition, we want to announce some of our findings using such programs (among them a confirmation of the result quoted above) as well as a rather general recursive enumeration formula which can be used to check such programs, but may be also of some interest of its own.

1. THE DEFINITION OF TILINGS

Let us recall that a tiling of $\mathbb{E}$ can be defined as a closed connected subset $T$ of $\mathbb{E}$ such that all connected components of $\mathbb{E} \setminus T$ are bounded and for any point $x \in T$ there exists a neighbourhood $U$ of $x$ and a natural number $r(x) = r_T(x) \geq 1$ such that the triple $(U, U \cap T, \{x\})$ is homeomorphic to the
triple.

\[(B^2 := \{ z \in \mathbb{C} \mid |z| \leq 1 \}, \{ z \in B^2 \mid z^{(a)} \in \mathbb{R}_+ \}, \{0\})\]

with \(\mathbb{R}_+ := \{ a \in \mathbb{R} \mid a \geq 0 \}\), as usual.

Two such tilings \(T, T' \subseteq \mathbb{E}\) are defined to be (topologically) isomorphic if there exists a homeomorphism \(\alpha\) of \(\mathbb{E}\) with \(\alpha(T) = T'\) in which case any such \(\alpha\) is defined to be a (topological) isomorphism from \(T\) to \(T'\), while the group of homeomorphisms \(\alpha\) of \(\mathbb{E}\) with \(\alpha(T) = T\), i.e. the group of automorphisms of \(T\), is denoted by \(\text{Aut}(T)\).

A tiling \(T\) is defined to be balanced if there exist two positive real numbers \(R_1\) and \(R_2\) such that any connected component \(f \in r_{\infty}(\mathbb{E} \setminus T)\) of \(\mathbb{E} \setminus T\) contains a full disc \(D(x; R_1) := \{ y \in \mathbb{E} \mid ||x, y|| \leq R_1 \}\) of radius \(R_1\) around some appropriate \(x \in f\) and is contained in such a disc of radius \(R_2\).

One says that \(T\) is cellular if the 'boundary' \(\partial f\) of any such \(f \in r_{\infty}(\mathbb{E} \setminus T)\) is homeomorphic to \(\partial B^2 = S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}\) while it is said to be pseudo-convex if it is cellular and if, in addition, for any two such connected components \(f_1, f_2 \in r_{\infty}(\mathbb{E} \setminus T)\) of \(\mathbb{E} \setminus T\) the intersection \(f_1 \cap f_2\) is connected. Also, \(T\) is defined to be normal if it is balanced and pseudo-convex. Note that two (topologically) isomorphic tilings are either both cellular (or pseudo-convex) or both non-cellular (or non-pseudo-convex), while one of them may be balanced without the other one being balanced. Hence we define a tiling to be topologically balanced (or normal) if it is isomorphic to a balanced (or normal) tiling.

2. The vertices of a tiling

For a tiling \(T\) of \(\mathbb{E}\) let \(\hat{T}\) denote the set of elements \(x\) of \(T\) with \(r(x) \neq 2\). It follows immediately from the above definitions that \(\hat{T}\) is a discrete subset of \(\mathbb{E}\), that any connected component \(e \in r_{\infty}(T \setminus \hat{T})\) of \(T \setminus \hat{T}\) is homeomorphic to the open interval \((0, 1)\) and that any such homeomorphism \(\varphi: (0, 1) \to e\) can be extended (uniquely) to a continuous map \(\hat{\varphi}: [0, 1] \to \hat{e} \subseteq e \cup \hat{T} \subseteq T\). This remains true if \(\hat{T}\) is replaced by any discrete subset \(T_0\) of \(T\), containing \(\hat{T}\). Hence we may specify any such subset \(T_0 \subseteq T\) as the set of vertices of our tiling \(T\). If \(T_0 = \hat{T}\), we say that \(T\) has no superfluous vertices and otherwise, of course, that it has such superfluous vertices. It depends very much on the context whether it makes sense to include tilings with superfluous vertices into the considerations. But even if one wants to restrict one's attention to tilings without superfluous vertices only, tilings with superfluous vertices will occur naturally while performing certain standard constructions by which one derives tilings from tilings. Hence we would like to introduce the following convention: if we talk about a tiling \(T \subseteq \mathbb{E}\)