ON $a$-SEMIAFFINE PLANES WITH INVISIBLE LINES

1. Introduction

A linear space is an incidence structure $L$ of points and lines such that any two points are on a unique line. The linear space $L$ is called an $a$-semiaffine plane (see [2]) if the following conditions are satisfied:

(i) For any non-incident point-line pair $(P, l)$, the number of lines through $P$ which do not intersect $l$ is either $a$ or $a - 1$.
(ii) Every line has at least two points and there are at least two lines.

These structures have already been studied in some detail. The following theorem summarizes what is known about finite $a$-semiaffine planes.

RESULT. Let $L$ be a finite $a$-semiaffine plane ($a \geq 1$).

(a) If $a = 1$, then $L$ is a (possibly degenerate) projective plane, an affine plane, or a linear space obtained by adjoining an infinite point to an affine plane (theorem of Dembowski and Kuiper, see Dembowski [3]; cf. also Beutelspacher [1, 9.3.9]).

(b) If $a = 2$, then, apart from one isolated case, $L$ is an affine plane, a punctured affine plane, an affine plane minus a line (Oehler [5]).

(c) If $a = 3$ and there is a point incident with at least 8 lines, then $L$ is obtained from a projective plane by removing the points of three non-concurrent lines or $L$ is a (hypothetical) 2-(46, 6, 1)-design (Beutelspacher and Meinhardt [2]).

(d) For any $a \geq 4$, there is at most a finite number of $a$-semiaffine planes (Beutelspacher and Meinhardt [2]).

The aim of this paper is to study linear spaces in which (i) holds only for some lines. In order to state our results, we have to introduce some notation. Let $L$ be a finite linear space. The number of points on a line $l$ is called the degree $k_l$ of $l$. Dually, the degree $r_P$ of a point $P$ is the number of lines through $P$; a point of degree $i$ is also called an $i$-point. For a non-
incident point-line pair \((P,l)\) we denote by \(\pi(P,l)\) the number of lines through \(P\) which do not intersect \(l\). Clearly, \(\pi(P,l) = r_P - k\). If \(n + 1\) is the maximal point degree of \(L\), then \(n\) is called the order of \(L\).

Suppose that the line set of a linear space \(L\) is partitioned into two sets, the ‘visible’ and the ‘invisible’ lines. Then \(L\) is called a weakly \(a\)-semi affine plane, if the following conditions hold:

(i) For any non-incident point-line pair \((P,l)\) of \(L\) we have \(\pi(P,l) \geq a - 1\); moreover, \(\pi(P,l) \leq a\) if and only if \(l\) is visible.

(ii) Every visible line has at least two points (‘good’ points) through which there are only visible lines. There are at least two visible lines.

We recall a definition of [2]. The linear space \(S_{n+1}\) consists of \(n + 2\) points and has exactly one 3-line, all other lines being 2-lines.

Our main results are the following two theorems.

**THEOREM 1.** Let \(L\) be a finite weakly \(a\)-semi affine plane \((a \geq 1)\) of order \(n\). Then every point has degree \(n + 1\) or \(n\) and every visible line has degree \(n + 2 - a\) or \(n + 1 - a\). If, furthermore, the following condition (*) holds:

\((*)\) there exists a good \((n + 1)\)-point

and there is a point of degree \(n\), then either \(L = S_{n+1}\), or \(a = 1\) and \(L\) is obtained from an affine plane by adjoining an infinite point.

**THEOREM 2.** Let \(L\) be a weakly 1-semi affine plane of order \(n\). Suppose that \(L\) satisfies (*). Then \(L\) is either a semi affine plane (hence known in view of the Dembowski–Kuiper theorem) or \(L\) is a projective plane of order \(n\) from which at most \(n\) points have been removed.

In other words, Theorem 2 says that one can characterize such structures if one considers only the visible lines. We also observe that in the case where there are no invisible lines, Theorem 2 reduces to the Dembowski–Kuiper theorem. It should be mentioned though that the proof of our theorem uses the Dembowski–Kuiper theorem.

In the next section we shall prove the above theorems. Then we shall present some examples, which will show in particular the following assertions:

- For any \(a \geq 1\) there is an infinite number of weakly \(a\)-semi affine planes. In particular, the analogous assertion of Result(d) does not hold.
- The embedding mentioned in Theorem 2 cannot be described in greater detail.