SIMPLICIAL CROSS-POLYTOPIC d-ARRANGEMENTS

ABSTRACT. Branko Grünbaum has observed that the projective $d$-arrangements formed by the facet hyperplanes of a regular polytope together with some of its hyperplanes of mirror symmetry and possibly the hyperplane at infinity are sometimes simplicial.

We investigate the projective $d$-arrangements associated in this manner with a cross-polytope. Fourteen such simplicial arrangements are known: three 1-arrangements, four 2-arrangements, six 3-arrangements, and one 4-arrangement. In this paper we prove that no other arrangement so associated with a cross-polytope is simplicial.

1. INTRODUCTION

A finite set $\mathcal{A}$ of hyperplanes in real projective $d$-space $\mathbb{P}^d$ is a (projective) $d$-arrangement when $\bigcap (\mathcal{A}) = \emptyset$. A $d$-arrangement is simplicial if each component of $\mathbb{P}^d \setminus \bigcup (\mathcal{A})$ is a simplex. The basic references on these matters are Grünbaum [3] and Grünbaum and Shephard [5]; see also Zaslavsky [6].

Grünbaum has suggested that when $\mathbb{E}^d$ is extended to $\mathbb{P}^d$ by adjoining a hyperplane at infinity in the classical manner, the projective $d$-arrangements obtained by including with the facet hyperplanes of a regular polytope in $\mathbb{E}^d$ some of its hyperplanes of mirror symmetry, possibly including the hyperplane at infinity, might be simplicial. (See page 56 of [3], page 3.5 of [4], or page 62 of [5].) In this paper we present an investigation of the simplicial $d$-arrangements that are associated in this way with a cross-polytope.

The case in which all of the hyperplanes of mirror symmetry and the hyperplane at infinity are included was resolved in [1], where we showed that the resulting $d$-arrangement is simplicial precisely for $d \leq 4$. Here we study the more general case.

Fourteen such simplicial arrangements are known. The four 2-arrangements $A_2(6), A_2(7), A_2(8)$, and $A_2(9)$ described on page 78 and pictured on page 87 of Grünbaum’s Catalogue of Simplicial Arrangements (pages 75–106 of [3]) are of this kind, as are the six 3-arrangements $A_3(12), A_3(13), A_3(14), A_3(15), A_3(17)$, and $A_3(18)$ described on pages 68–82 of Grünbaum and Shephard [5] (although this is not obvious for $A_3(13), A_3(14)$, or $A_3(15)$; see the proof of Theorem 5 below). And in [1] we described a 4-arrangement $A_4(33)$ of this kind.

Our objective here is to prove that, apart from three trivial 1-arrangements, no other $d$-arrangement so associated with a cross-polytope is simplicial.

We begin in Section 2 with a few preliminaries concerning $d$-arrangements in general and cross-polytopic $d$-arrangements in particular. In Section 3 we consider the cases $d \leq 3$, and in Section 4 we examine $d = 4$. The situation for $d \geq 5$ is settled in the concluding Section 5.

It is a pleasure to express our gratitude to Branko Grünbaum for his interest in our results.

2. Cross-polytopic $d$-arrangements

In this section we collect some of the basic facts and notations we shall employ throughout.

Write $\infty^d$ (or just $\infty$) for the hyperplane at infinity. It will be convenient to take the $r$-cells formed by a $d$-arrangement $\mathcal{A}$ to be closed and to call such a cell $F'$ bounded, finite, or unbounded according to whether $F' \cap \infty^d = \emptyset$, $\dim(F' \cap \infty^d) < r$, or $\dim(F' \cap \infty^d) = r$. These definitions reflect our essentially Euclidean point of view.

If $L'$ is any $r$-flat in $\mathbb{P}^d$ with $0 < r \leq d$ and $\mathcal{A}$ is a $d$-arrangement, the $r$-arrangement $\mathcal{A}|_{L'}$ induced by $\mathcal{A}$ in $L'$ (regarded as a $\mathbb{P}^r$) is the arrangement

$$\mathcal{A}|_{L'} = \{ F' \subseteq L' : F' \in \mathcal{A}, \dim(F' \cap L') = r - 1 \}.$$

Since for $1 \leq r \leq s$, every $r$-face of an $s$-simplex is an $r$-simplex, the following useful result is obvious.

**Lemma 1.** The $r$-arrangement $\mathcal{A}|_{L'}$ induced by a simplicial $d$-arrangement $\mathcal{A}$ in each of its $r$-flats $L'$, $1 \leq r \leq d$, is simplicial.

As a practical matter, to describe $\mathcal{A}|_{L'}$ it is generally easiest to obtain equations of the hyperplanes ($(r - 1)$-flats) of $\mathcal{A}|_{L'}$ by parametrizing $L'$.

We turn next to cross-polytopic arrangements. The cross-polytope $C^d$ in $\mathbb{R}^d$ is the convex hull of the $2d$ unit points whose coordinates are the permutations of $(\pm 1, 0, \ldots, 0)$. Basic facts about $C^d$ can be found in Coxeter [2]; see Chapter VII, especially pp. 121, 122, 133.

Introduce homogeneous coordinates $(x_1, \ldots, x_d; x_0)$ in $\mathbb{P}^d$, with $x_0 = 0$ the hyperplane $\infty^d$ at infinity. Then the $2^d$ facet hyperplanes of $C^d$ have homogeneous equations

$$\pm x_1 \pm x_2 \pm \cdots \pm x_d = x_0. \tag{1}$$

The $d^2$ hyperplanes of mirror symmetry are of two kinds: there are $d$ coordinate hyperplanes

$$x_i = 0, \quad 1 \leq i \leq d, \tag{2}$$

each of which contains all but two of the $2d$ vertices of $C^d$ (from which it is equidistant), and there are $d(d - 1)$ hyperplanes with equations

$$x_i \pm x_j = 0, \quad 1 \leq i < j \leq d,$$

each of which lies midway between two parallel opposite edges of $C^d$ (from