A COMBINATORIAL CHARACTERIZATION OF
THE DUAL MOUFANG HEXAGONS

1. Introduction

In [3], J. Tits announces a classification of all Moufang hexagons (i.e. generalized hexagons satisfying the Moufang condition, described in Section 2.4), and in [2] I show that the 'usual' Moufang hexagons are characterized by certain sets of points, called ideal lines. In this paper the 'dual' Moufang hexagons (see Section 2.5) are characterized by certain sets of points called intersection sets, together with a geometric condition called the regulus condition (see Section 2.3). The main theorem is as follows.

THEOREM 1.1. If \( H \) is a generalized hexagon with the regulus condition, then:

(i) all intersection sets have only one point if and only if \( H \) is a 'usual' Moufang hexagon.
(ii) all intersection sets have at least two points if and only if \( H \) is a 'dual' Moufang hexagon.

Remark. Every Moufang hexagon is either 'usual' or 'dual'; for the distinction between the two types see Section 2.5.

COROLLARY 1.2. If \( H \) is a finite generalized hexagon with the regulus condition, then all intersection sets have one point if and only if \( H \) is the usual \( G_2(q) \) or \( 3D_4(q) \) hexagon, two points if and only if \( H \) is the dual \( G_2(q) \) hexagon, and \( q^2 + 1 \) points if and only if \( H \) is the dual \( 3D_4(q) \) hexagon.

Remark. Tits has shown that every Moufang hexagon is associated with a simple algebraic group or a mixed group of type \( G_2 \); a complete list is given in [3] and in [2]. In particular every finite Moufang hexagon is associated with one of the groups \( G_2(q) \) or \( 3D_4(q) \).

2. Definitions and notation

2.1. If \( H \) is a generalized hexagon we let \( d: H \times H \to \{0, 1, \ldots, 6\} \) be the distance metric on the flag-graph of \( H \), and we call two points (or lines) \( x \) and \( y \) opposite if \( d(x, y) = 6 \). In case \( d(x, y) = 4 \) we let \( x \ast y \) denote the unique point (or line, if \( x \) and \( y \) are lines) collinear (or concurrent) with both \( x \) and \( y \). If \( d(x, y) = 6 \) we let \( x \ast y \) be the set of lines (or points) distant 3 from both \( x \) and \( y \), i.e. \( x \ast y = \Gamma_3(x) \cap \Gamma_3(y) \), where \( \Gamma_4(x) = \ldots \)
\{h \in \mathcal{H} \mid d(x, h) = i\}. If \(p\) and \(q\) are collinear points we let \(pq\) denote the unique line through \(p\) and \(q\).

If \(p\) and \(q\) are opposite points of \(\mathcal{H}\) we write \(p^\# = \Gamma_2(p) \cap \Gamma_4(q)\), i.e. those points collinear with \(p\) which are not opposite \(q\); there is precisely one point of \(p^\#\) on each line through \(p\). Similarly one defines \(L^M\) for lines \(L\) and \(M\) of \(\mathcal{H}\) which are opposite.

2.2. Intersection Sets

If \(x\) and \(y\) are two points opposite \(z\), such that \(d(x, y) = 4\) and \(d(z, x \ast y) = 4\) (see Figure 1 in Section 3.1), then we call \(z^x \cap z^y\) an intersection set if \(z^x \neq z^y\). Since \(w = (x \ast y) \ast z \in z^x \cap z^y\) we see that \(z^x \cap z^y\) always contains at least one point.

2.3. The Regulus Condition

If \(x\) and \(y\) are opposite (points or lines), then we say that \(\mathcal{H}\) has the regulus condition if \(a \ast b = a \ast c\) for all distinct \(a, b, c \in x \ast y\). In this case we write \(R(x, y) = a \ast b\), where \(a, b \in x \ast y\), and call it the regulus spanned by \(x\) and \(y\). This terminology was suggested by W. M. Kantor in view of the fact that if \(L\) and \(M\) are opposite lines in the usual \(G_2(k)\) hexagon, then they can be thought of as lines in the orthogonal geometry of a 7-dimensional vector space, in which \(R(L, M)\) is a regulus in the sense of Dembowski [1, p. 220]. It is easily seen that the regulus condition is self-dual.

2.4. Apartments and Root Groups

The following terminology is due to J. Tits. An ordinary hexagon (circuit of six points and six lines) in \(\mathcal{H}\) is called an apartment, and a chain of length 6 (in the flag-graph) is called a root. If \(x\) is a root we let \(\partial x\) denote the end points (or lines) of \(x\), and we let \(U_x\) denote the subgroup of \(\text{Aut} \mathcal{H}\) fixing all points and lines incident with a line or point of \(\partial x\). \(U_x\) is called a root group; it acts semi-regularly on apartments containing \(x\) (equivalently, on lines (or points) incident with a point (or line) of \(\partial x\)), see [2; Section 2], and \(\mathcal{H}\) is said to be Moufang if all \(U_x\) are transitive on apartments containing \(x\).

If \(\Sigma\) is an apartment, there are twelve distinct roots contained in \(\Sigma\), each uniquely determined by its middle element (point or line) \(v \in \Sigma\). We let \(\Sigma_v\) denote the unique root in \(\Sigma\) with middle element \(v\), and we call it a long (resp. short) root if \(v\) is a line (resp. point) of \(\mathcal{H}\); accordingly one has long and short root groups. Clearly long roots of \(\mathcal{H}\) are short roots of \(\mathcal{H}^*\) and vice versa. If \(p\) and \(L\) are incident we say \(\Sigma_p\) is adjacent to \(\Sigma_L\).