VARIANGULAR WIND SPIRALS

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(Received 10 October, 1969)

Abstract. The term 'variangular' is introduced to emphasize a significant difference between the present and certain earlier solutions to the problem of organized airmotion within the planetary boundary layer. The latter belong to the family of equiangular wind spirals and have the characteristic that the angle ($\theta$) formed by the vectors of shearing stress and geostrophic departure is invariant with height; it is shown that in this spiral-family, parabolic height-dependency of the effective (eddy) diffusivity ($K$) alone is permitted, including the asymptotic case of constant $K$; the famous Ekman spiral as well as the Rossby spiral are two prominent members of the family of equiangular wind spirals. The new variangular theory, as the name implies, permits variation of $\theta$ with height ($z$) and produces more versatile profiles of wind and stress due to less restraint in $K(z)$. As an example of comparison with observed data, monthly mean wind profiles obtained at Plateau Station, Antarctica, are selected since they exhibit a noteworthy degree of 'variangularity', in relatively satisfactory agreement with properties of the new theoretical model for wind spirals.

1. Introduction

Lettau (1962a, b) has investigated and developed two types of theoretical models of wind spiral structure. The first can be referred to as the 'l-theory' since it is based on a universally specified height-variation of the turbulent length-scale ($l$) for laboratory as well as natural boundary layer flow. In the atmosphere it pertains to given values of the Coriolis parameter ($f$), aerodynamic surface roughness ($z_0$), and horizontal pressure gradient or geostrophic speed ($V_g$); the sole scaling-factor is the surface Rossby number ($V_g/z_0 f$); solution was achieved numerically by the Cauchy-method, yielding Prandtl's logarithmic wind profile formula asymptotically for the surface layer. For another example of 'l-theory' see Blackadar (1962). The second type mentioned above can be referred to as the 'K-theory' since it is based on an implicitly specified height-variation of effective (eddy) diffusivity ($K$); explicitly, it considers the general characteristics of angular variables as functions of height which determine a wind spiral, but these are uniquely related to $K(z)$. Previously, treatment by 'K-theory' has normally been restricted to monotonical height dependency which causes unrealistic results. For earlier examples with explicitly specified power-laws for $K$ see Koehler (1932), or Prandtl and Tollmien (1925).

One objective of spiral research must be to provide a simple and practical, yet reliable and realistic, method for estimating certain derived flow parameters (e.g.,

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eddy diffusivity for atmospheric diffusion problems, or shearing stress for problems of energetics, etc.) from directly observable variables (e.g., wind velocity, wind shear including rate of veering and turning) within the atmospheric boundary layer. If ‘K-theories’ are to be used, the variangular model of spiral structure is a step in this direction since it yields more realistic results than conventional spiral models.

2. Generalized Spiral Solution

The formulation is restricted to steady and horizontally uniform large-scale motion within the planetary boundary layer. It is assumed that at any point in the flow there exists a perfect balance among Coriolis, friction and pressure gradient forces; inertia forces and mean vertical components are thus excluded. By definition, the friction force goes to zero at the top of the boundary layer and the vector of mean motion ($\mathbf{V}$) approaches the geostrophic value. Barotropy is assumed; i.e., thermal winds are excluded and the vector of geostrophic wind ($\mathbf{V}_g$) is independent of height. In practice, thermal wind contributions to the motion may be eliminated and the data then analyzed to determine the effects of stability on the wind and stress distribution. Vertical variations of density ($\rho$) are dismissed as negligibly small because the layer under consideration is relatively shallow in comparison with the thickness of the entire troposphere.

Under the imposed restrictions, the equation of motion reduces to

$$f_i \rho (\mathbf{V} - \mathbf{V}_g) = (\tau)_z,$$

where complex vector notation is used ($i = \sqrt{-1}$), the subscript $z$ (and only $z$) denotes partial differentiation with respect to height, and the Coriolis parameter is given by $f$. The vector of shearing stress ($\tau$) is assumed horizontal and proportional to the wind shear vector. The factor of proportionality defines the effective dynamic viscosity ($\rho K$), or the eddy diffusivity ($K$). Upon recalling the constancy of $\mathbf{V}_g$, the shearing stress may be expressed in terms of the geostrophic departure ($\mathbf{V} - \mathbf{V}_g$),

$$\tau = \rho K (\mathbf{V} - \mathbf{V}_g)_z.$$

Equations (1) and (2) express the basic nature of the problem and may be combined to yield an interdependent set of two second-order, nonlinear homogeneous differential equations of the form

$$f_i (\mathbf{V} - \mathbf{V}_g) = K (\mathbf{V} - \mathbf{V}_g)_{zz} + K_z (\mathbf{V} - \mathbf{V}_g)_z,$$

$$f_i \tau = K (\tau)_{zz}.$$

The nonlinearity of Equations (3) and (4) arises from the fact that the eddy diffusivity (cm$^2$ sec$^{-1}$) is the product of a length and a velocity; while the length term may be some universal and explicit height function, the velocity term is known to depend on the value of the shearing stress. The latter, in turn, is a dependent variable in the basic equations and hence the problem is seen to be inherently nonlinear. This fact needs