THE APPLICATION OF DIMENSIONAL ANALYSIS TO COSMOLOGY
(or, How to Make Cosmology Simple by Using Dimensional Conspiracy)

PAUL S. WESSON
Institute of Theoretical Astrophysics, University of Oslo, Norway
and
Dept. of Physics, University of Alberta, Canada

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Abstract. Cosmology as it is usually studied suffers from the problem that no criterion is known which isolates from the large class of models allowed by the equations of physics those few which are realized in Nature. To provide such a criterion, it is proposed that cosmology should be based on the study of models which are free of arbitrary scales or units, this condition being compatible with (but not identical with) the Cosmological Principle. Formally, the basis for scale-free cosmology can be expressed in a dimensional Conspiracy Hypothesis: The material parameters of a system (mass, density, pressure etc.), the constants of physics and the coordinates have realizable physical meanings only when they occur together in dimensionless combinations (\(\eta\)-numbers) in which the components may vary with time or place but in such a manner that the variations conspire to keep the \(\eta\)-numbers constant. The Conspiracy Hypothesis (CH) streamlines cosmology, simplifying it to the finding of a few dimensionless numbers. Applied to Einstein's general relativity, the CH yields a simple cosmological model consisting of static clusters of galaxies with inverse-square density profiles embedded in an expanding, homogeneous background. This model agrees well with the observed Universe insofar as the latter can be described by general relativity. The CH can also be applied to other theories of gravity, especially those in which the gravitational parameter \(G\) is variable, and can also in itself be taken as a basis for gravitational theory.

1. Introduction

The object of this article is to try to understand why the Universe is relatively simple in structure rather than being complicated. Another way of stating this aim is to pose the question: Why is the Universe as realized in Nature more simple than the (mostly) complicated solutions allowed by the equations of physics?

To answer this question, the proposal is made on theoretical grounds that the Universe should be simple because it should be described by physics that is free of arbitrary (human-invented) scales of length, time and mass. This criterion of being scale-free is formalized both in words and symbols by a Conspiracy Hypothesis. This name is suggested by a rather intriguing property of the solutions to which it leads: many physical parameters whose variations we measure with the aid of human-invented scales in the laboratory have variations in cosmology which are hidden from us by virtue of conspiracy, the only data which have well-defined meanings in cosmology being certain constant dimensionless numbers. An application of the Conspiracy Hypothesis to the equations of general relativity is found to yield a model which is in good agreement with the observed Universe. This suggests that conspiracy is not merely an interesting property of some solutions of the equations of gravitational physics, but is also a property of the actual Universe.

The present account is meant to be preliminary. Nevertheless, in introducing a hypothesis like conspiracy it is necessary to show how it relates to other principles.
which are widely regarded as forming a foundation for cosmology. This is especially true of the Cosmological Principle. For this reason, Section 2 examines at some length how the concept of scales (and their absence) relates to the Cosmological Principle. The motive behind this section is to present the case for taking the absence of arbitrary scales to be a workable basis for cosmology. Those who are prepared to adopt this belief without argumentation may skip Section 2. Having (it is hoped) justified the scale-free criterion, the formal statement of conspiracy is given in Section 3, and applied to general relativity in Section 4. The discussion of Section 5 deals briefly with some further lines of development of the conspiracy idea, and suggests ways in which the Conspiracy Hypothesis can be further tested. Section 6 is a conclusion, while mathematical details relevant to the results of Sections 2 to 5 are relegated to the Appendix.

2. Scales and the Cosmological Principle

Most of the laws of physics are expressed in the form of partial differential equations. To obtain a mathematical description of a given physical system, what is usually done is to (i) make assumptions about what physical properties of the system are important, (ii) integrate the equations subject to these assumptions and then (iii) fix any arbitrary (integration-produced) functions of the coordinates which may have entered by adopting a set of boundary conditions. Contact with reality occurs at stages (i) and (iii), and in this regard the last stage is at least as important as the first one because boundary conditions are often chosen to agree with observation. Boundary conditions, or other conditions of constraint, have considerable logical importance because they usually involve the introduction, either explicitly or implicitly, of scales. These scales are often simple quantities consisting of one or more of the fundamental parameters mass, length and time – in terms of which experience has taught us that all mechanical quantities may be expressed. These scales, besides expressing conditions of physical constraint, also provide a set of units (e.g., gm, cm, sec). With a set of units, physical parameters other than simple scales may be expressed in convenient form (e.g., the density has the dimension $[ML^{-3}]$ or the units gm cm$^{-3}$). The concepts of scale and unit are closely connected. [Birkhoff (1950), Langhaar (1951), and Sedov (1959) have discussed these concepts. The mathematical basis of dimensional analysis as a whole is the Pi Theorem, which basically says that ‘dimensionless’ = ‘unit free’: see Birkhoff, 1950, pp. 82–85.] From the point of view of logic, the reason for the existence of scales and units in physics lies in the existence of boundary conditions or other conditions of constraint.

The methodology discussed in the preceding paragraph is all right when applied to nearly all physical problems. For example, the existence of a tank size ($L_0$) in a problem of laboratory hydrodynamics; the existence of a characteristic period ($T_0$) in the problem of tides on the Earth; or the existence of a stellar mass ($M_0$) in the problem of planetary motion – all can be studied in terms of the methodology