Thermomechanical States

with Controllable Invariant Relations

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Communicated by I. MÜLLER

1. Introduction

In previous work [5] I have shown that constitutive information derived from experiments based on controllable states of elastic heat conductors, although necessarily incomplete, may be applied to the study of certain problems in nonlinear thermoelasticity. The thermomechanical states presented in that work all involve deformations with constant strain invariants coupled with quite general temperature fields in cylindrical and spherical geometries, and some special cases of them have subsequently been shown to be controllable in a restricted class of heat conductors [6].

In the present work I derive thermomechanical states involving large deformations possessing non-constant strain invariants and general temperature fields that are associated with the same incomplete controllable-state data and find that various combinations of bending, inflating, extending, and straightening of blocks and annular wedges, coupled with temperature gradients in various directions, and the inflation of spherical shells, with radial heating, involve invariant combinations of strain and temperature gradients that are related in exactly the same way as those of the controllable states. Therefore, these associated states may be analyzed to varying degrees with the partial constitutive data derived from controllable states to determine their possibility or to calculate the body force and heat generation fields necessary to effect them in given thermoelastic materials.

The extent to which such analyses may be carried out depends on the thermomechanical state and the material properties as expressed in the controllable-state data. In the case of constant strain invariants it has been shown [5] that one may be able to construct new exact solutions for special classes of materials or solve problems in nonlinear thermoelasticity with heat generation in closed form under certain conditions. Since the working out of such examples is routine, albeit more complicated when the strain invariants are not constant, I shall not present any examples here.

The procedure for constructing the associated states is straightforward. After noting the orientation that the temperature gradient must have with respect to the directions of principal stretch for various kinds of degeneracy, I am able to
construct the most general deformation tensor possible under certain assumptions. The vanishing of the Riemann tensor associated with the strain tensor serves as an integrability condition, and, finally, I integrate sets of compatible strain-displacement equations. In order to delimit the cases it is necessary to consider, I restrict the analysis by treating only cases where (1) the surfaces of constant temperature and strain invariants are parallel planes, coaxial circular cylinders, or concentric spheres, though the temperature and strain invariants may be constant on orthogonal families of these surfaces and (2) the directions of principal stretch coincide with rectangular Cartesian, circular cylindrical, or spherical polar coordinate lines. Such restrictions enable me to present a complete analysis in a reasonable amount of space.

The scope of the analysis involves the standard geometries encountered in thermal-stress-analysis problems, and the restrictions are ones that naturally arise in many interesting boundary-value problems. The potential usefulness of controllable-state constitutive information is not limited to analyzing states with these restrictions, however. It has been shown [4], e.g., that the sphering of an annulus, accompanied by a radial or latitudinal temperature gradient, has associated with it invariants which are not constant on planes, cylinders, or spheres and yet have the relations possessed by the controllable-state invariants.

Such more complicated states, with invariants depending on more than a single curvilinear coordinate, cannot be expected to solve the field equations as nearly as do the states constructed in the present work. Indeed, several of the states constructed can be shown to satisfy one or the other of balance of momentum or balance of energy identically in the material response functions, thereby making these states good foundations for approximate solutions to problems in nonlinear thermoelasticity.

After putting down the kinematical, field, and constitutive equations necessary for the work, I present the pertinent controllable-state information in Section 4 and then construct the thermomechanical states associated with that information for distinct principal stretches, in Section 5, and for two types of degeneracy, in Sections 6 and 7. The states found analyzable with controllable state data are summarized in Section 8 below.

The notation is that of general tensor analysis. Ad hoc conventions important to Sections 5, 6, and 7 are presented at the beginning of Section 5. The reader is referred to Truesdell & Noll’s article [2] in the Handbuch der Physik for background to the equations of the nonlinear field theories that are employed.

2. Kinematics

We use \( x^i \) and \( X^J \) to denote curvilinear coordinates of the positions occupied by material particles in the deformed and undeformed configurations, respectively, of a continuous body subject to the static deformation \( x^i = x^i(X^J) \). The left Cauchy-Green tensor is an appropriate strain measure, and the components \( (c^{-1})^{ij} \) of it and \( c_{ij} \) of its inverse are given by

\[
(c^{-1})^{ij} = G^{JK} \frac{\partial x^i}{\partial X^J} \frac{\partial x^j}{\partial X^K},
\]

(2.1)