On the Representation Theorems of Poisson, Riemann and Volterra for the Euler-Poisson-Darboux Equation

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1. Introduction

This paper continues our study of the representation of solutions of the Euler-Poisson-Darboux equation [4]. This linear partial differential equation, of the hyperbolic type in two independent variables, has the form

\[ \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2\alpha}{r} \frac{\partial \phi}{\partial r} \]  

where we have assumed \( \alpha \) to be a real parameter such that \( 0 < \alpha < 1 \). For a discussion of other real values of \( \alpha \) we refer to [4] and [9]. This equation has been the object of much study, in part due to its interest in many specific problems in classical physics [3] and in part due to its intrinsic mathematical interest [9]. On the latter points, it is perhaps the most simple linear partial differential equation with a singular line \((r = 0)\). As we have remarked in [4], the presence of this line poses certain questions which are beyond the scope of the standard theory.

There are three different representations known for equation (1.1), the first one having been given by Poisson [6] in 1823. It has the form

\[ \phi(t, r) = \int_0^\pi f(t + r \cos \psi) \sin^{2\alpha - 1} \psi d\psi + r^{1 - 2\alpha} \int_0^\pi g(t + r \cos \psi) \sin^{1 - 2\alpha} \psi d\psi \]

for \( 0 < \alpha < 1 \), \( \alpha \neq 1/2 \). For \( 0 < \alpha < 1/2 \), the first integral dominates in the limit \( r \to 0^+ \), while for \( 1/2 < \alpha < 1 \), the second one does. Given that \( f(\tau) \) and \( g(\tau) \in C^2 \) in the interval \( t - r \leq \tau \leq t + r \), we can show that (1.2) satisfies (1.1) and that

\[ \lim_{r \to 0^+} \phi(t, r) = \frac{\Gamma(1/2) \Gamma(\alpha)}{2 \Gamma(\alpha + 1/2)} f(t) \]

and

\[ \lim_{r \to 0^+} r^{2\alpha} \frac{\partial \phi}{\partial r} = \frac{\Gamma(1/2) \Gamma(1 - \alpha)}{\Gamma(1/2 - \alpha)} g(t) \]

provided that \( 0 < \alpha < 1/2 \). If \( 1/2 < \alpha < 1 \), these limits can be read as finite parts in the sense of Hadamard. The two integrals in (1.2) are clearly linearly independent save for the case \( \alpha = 1/2 \).
For the exceptional case $\alpha=1/2$ we write (1.2) in the form (following Darboux [2])

$$
\phi(t, r) = \int_0^\pi \left[ \frac{f(t + r \cos \psi)}{2} - \frac{g(t + r \cos \psi)}{1 - 2\alpha} \right] \sin^{2\alpha-1} \psi \, d\psi
$$

(1.2a)

$$
+ r^{1-2\alpha} \int_0^\pi \left[ \frac{f(t + r \cos \psi)}{2} + \frac{g(t + r \cos \psi)}{1 - 2\alpha} \right] \sin^{1-2\alpha} \psi \, d\psi.
$$

In the limit $\alpha \to 1/2$, (1.2a) becomes

$$
\phi(t, r) = \int_0^\pi f(t + r \cos \psi) \, d\psi + \int_0^\pi g(t + r \cos \psi) \ln(r \sin^2 \psi) \, d\psi
$$

so that in this case we have

$$
\lim_{r \to 0^+} \phi(t, r) = \pi f(t) \quad \text{and} \quad \lim_{r \to 0^+} r \frac{\partial \phi}{\partial r} = \pi g(t).
$$

A second representation is to be found for the special case $\alpha=1/2$ in Volterra's work [8] of 1892. For $0 < \alpha < 1$, equation (1.1) is satisfied by either of the following representations:

$$(1.3) \quad \int_0^\infty \left[ f_1(t + r \cosh \psi) + f_2(t - r \cosh \psi) \right] \sinh^{2\alpha-1} \psi \, d\psi$$

or

$$(1.3a) \quad r^{1-2\alpha} \int_0^\infty \left[ f_3(t + r \cosh \psi) + f_4(t - r \cosh \psi) \right] \sinh^{1-2\alpha} \psi \, d\psi.$$ 

In order that (1.3) and (1.3a) be solutions of equation (1.1) we require order conditions on $f_1(\tau), f_3(\tau), f_2(-\tau)$ and $f_4(-\tau)$ as $\tau \to \infty$ as well as some smoothness conditions on these functions, and we shall discuss them in Section 3. Observe that (1.3) and (1.3a) employ data which are outside of the interval $(t-r, t+r)$. We have already discussed in [4], Section 8, the interpretation of $f_3(\tau) + f_4(-\tau)$. Incidentally, Lamb [5] felt that for the special case $\alpha=1/2$, (1.3) was preferable to (1.2) since (1.3) embraced both converging and diverging waves while (1.2) did not discriminate them. We shall see that this point is illusory.

The third representation was given by Riemann in 1860 in terms of what is now called a Riemann function [7]. In fact this one may be used to obtain $\phi(t, r)$ for $0 < t < r$ when the data $\phi(t, r)$ and $\frac{\partial \phi}{\partial t}$ are assigned at $t=0$. If, however, such data are given when $0 < t < r$, then there are some modifications in the results of Riemann, as Copson [1] had pointed out in 1958. Copson had made extensive use of certain integral transforms to find these modifications. The results of Poisson, Volterra and Copson are all equivalent in the restricted interval of $\alpha$ which we employ. It is, of course, possible to extend this interval by methods which are standard for equation (1.1) [9], but we shall not pursue this point here. It is to be noted that we do not require data beyond the point $r+t$ on the $t$ axis.

If $0 < r < t$, assumptions regarding the behavior of $\phi$ and $\frac{\partial \phi}{\partial t}$ at $t=0$ in the neighborhood of $r=0^+$ will be needed.