The Lower Spectrum of Schrödinger Operators

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Introduction

Let \( G \) be a domain in \( \mathbb{R}^n \), where \( n \geq 2 \). In the Hilbert space \( H = L^2(G) \) we consider the (minimal) Schrödinger operator \( T_0 \) defined by

\[
T_0 u = \tau u, \quad D(T_0) = \mathcal{D}_0,
\]

where

\[
\tau = -A + q, \quad \mathcal{D}_0 = C_0^\infty(G),
\]

and \( q: G \to \mathbb{R} \) is a locally Hölder continuous function. It is easy to prove that \( T_0 \) is semibounded, and even positive, if one can find a positive function \( f \in C^2(G) \) with \( \tau f \geq 0 \). Indeed the following identity then holds:

\[
\tau u = (f^{-1} \Delta_f) (f \Delta_f)^{-1} u + \left( q - \frac{4f}{f} \right) u, \quad u \in C^2(G),
\]

(1)

(where \( \Delta_f = (1/i) \nabla \cdot f \nabla \)) which immediately implies that

\[
(T_0 u, u) = \int_G \frac{|f|^2}{f} \left| \nabla \frac{u}{f} \right|^2 \, dx + \int_G \frac{\tau f}{f} |u|^2 \, dx \geq 0, \quad u \in \mathcal{D}_0.
\]

Each function \( f \in C^2(G) \) with \( \tau f = \lambda f \), \( \lambda \in \mathbb{C} \), is called an eigensolution of \( \tau \) corresponding to the eigenvalue \( \lambda \). If \( f \) is a positive eigensolution of \( \tau \) corresponding to \( \lambda = 0 \), then in (1) the second term vanishes and \( T_0 \) is formally represented as a Jacobi factorization aggregate \( \mathbf{B}^* \mathbf{B} \). Since this representation is very useful for many spectral-theoretic researches (cf. [2], [6], [10]), the question arises whether conversely the semiboundedness of \( T_0 \) implies the existence of a positive eigensolution. The answer is given by a theorem essentially due to ALLEGRETTO [1] (for a generalized version and a short proof see MOSS & PIEPENBRINK [7]).

**Theorem 1.** Let \( T_0 \) be semibounded and let \( T \) denote the Friedrichs extension of \( T_0 \). Let \( \lambda \in \mathbb{R} \). Then \( \lambda \leq \inf \sigma(T) \) if and only if there exists a positive eigensolution of \( \tau \) corresponding to \( \lambda \).

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This is the multidimensional analogue of a property that is clearly valid for the corresponding ordinary differential operator (cf. [10]). Still another proposition which is closely related to this theorem, and which likewise forms part of the oscillation theory of the multidimensional Schrödinger operator, is due to GLAZMAN and MOSS & PIEPENBRINK. In order to state this result we need the concept of the lower spectrum: Let $T$ be a selfadjoint operator and let $\lambda_0 \in (-\infty, \infty]$ be the infimum of its essential spectrum ($\inf \sigma = \infty$); the set $(-\infty, \lambda_0) \cap \sigma(T)$ is called the lower spectrum of $T$. Then we have ([4], p. 159, [7])

**Theorem 2.** Let $G = \mathbb{R}^n$. Let $T_0$ be semibounded* and let $\lambda_0 = \inf \sigma_{\text{ess}}(T) < \infty$. Then the lower spectrum of $T$ is finite if and only if there is an eigensolution of $T$ corresponding to $\lambda_0$ which is positive in the complement of some closed ball in $\mathbb{R}^n$.

It is the aim of this paper to show how one can employ these theorems to derive criteria about the size (non-empty, infinite) of the lower spectrum of Schrödinger operators (criteria about the finiteness of the spectrum, based on Theorem 2, can be found in ALLEGRETTO [1] and PIEPENBRINK [8]). For the sake of simplicity we consider only the case $G = \mathbb{R}^n$. We note first an easy consequence of the theorems above.

**Theorem 3.** Let $G = \mathbb{R}^n$ and suppose that $T_0$ is semibounded and $\inf \sigma_{\text{ess}}(T) = 0$. Then

(a) $T$ has at least one negative eigenvalue if and only if $T$ has no positive eigensolutions corresponding to $0$.

(b) $T$ has infinitely many negative eigenvalues if and only if $T$ has no eigensolution corresponding to $0$ which is positive in the domain $\{x : |x| > R\}$, for any $R > 0$.

Henceforth we suppose that the assumptions of Theorem 3 hold; this is the case if, for instance,

$$q(x) \to 0 \quad \text{as} \quad |x| \to \infty. \quad (2)$$

Then $T_0$ is semibounded and the essential spectrum of $T$ is just the interval $[0, \infty)$ (cf. REED & SIMON [9, p. 119]; in fact, an even stronger result is given there). The following result is an immediate consequence of Theorem 3; cf. also [7, Corollary, p. 225].

**Theorem 4.** Let $n = 2$. If $q \leq 0$ and $q \not= 0$, then $T$ has at least one negative eigenvalue.

To prove this, assume that $\tau$ has a positive eigensolution $f$ corresponding to $0$. Then $Af = qf \leq 0$ so that $f$ is superharmonic. Since $f$ is bounded below it must be constant according to Liouville’s theorem (see [3, p. 29]); this contradicts the condition $q \not= 0$.

The hypothesis of this theorem can be considerably weakened. We shall prove the following criterion (for the case $n = 1$, cf. [10]).

* Thus the closure $T$ of $T_0$ is selfadjoint according to WIENHOLTZ’S theorem, cf. SIMADER [11].