A Formal Description of Representation Theorems for Constitutive Functions

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I. Introduction

1. Apology

Representation of constitutive functions under the combined requirements of material objectivity and symmetry is a classical problem, well-discussed in the literature of continuum mechanics. The bulk of this literature is devoted to specific examples, to the presentation of particular representations for certain classes of arguments and types of symmetry. However involved the algebraic details of these papers it is not difficult to see a general pattern for the representation process; in this paper I abstract this pattern and present a formal general representation theorem. WANG, in [W], has given a complete solution of the general representation problem, but I feel that the representation procedure he describes is not completely characteristic of the usual procedure, and his proof uses the transformation-group structure of the problem. It seems to me, on the other hand, that the proper setting for discussion of representation theory is within function theory: the labor of representations must be in linear algebra and group theory but the form of the theory has to do only with equivalence relations and induced functions.

Thus what I do in this paper is to model the representation process in function-theoretic terms. Necessarily this involves an examination of the structure of the collection of equivalence relations on a set, which is presented in Section II: the focus is on the problem of combined equivalences, a topic which arises in a natural, albeit trivial, way in considering invariance under objectivity and symmetry and in a non-trivial way when one considers combinations of symmetries and which also is central to the constructions of the representation theorems of Section III.

I must point out that I do not present any new representations or labor-saving devices. At best I would describe this paper as presenting the proper point of view from which to consider the whole process of representation. To illustrate the usefulness of this point of view, in Section IV I use it to answer a
question raised by Truesdell & Moon [TM]: does an isotropic symmetric-tensor-valued function of a symmetric tensor have an isotropic inverse? I find that the answer to this question, in a generalized context, is always yes for injective maps and I present a necessary and sufficient condition that a surjective map have an isotropic inverse, as well as a necessary and sufficient condition for the existence of a semi-inverse [TM]. I also believe that this theoretical structure offers a more convenient context in which to consider interesting problems regarding the inheritance of smoothness or differentiability by the representative functions from the parent mapping.

2. Description

The constitutive equations of continuum mechanics are required to be invariant under change of observer, or orthogonal transformations of current placement, and may have symmetries, i.e., be invariant under certain groups of transformations of the reference placement. This is formalized as follows: we are given sets $S$ and $A$ and groups $Q$ (change of observer) and $G$ (symmetry transformation) acting, on the left and on the right respectively, upon both $S$ and $A$. We want to characterize those maps $\pi: S \rightarrow A$ having the property

\[ \forall s \in S, Q \in Q, G \in G : \pi(QsG) = Q\pi(s)G. \quad (1) \]

The standard representation theorem, which appears in generalized form as Theorem 36 below, then asserts that there exists a set $C$ and a map $\phi$, both determined solely by $S$, $A$, $Q$, $G$,

\[ \phi: S \times C \rightarrow A, \]

with the property that to each $\pi$ obeying (1) may be associated a map

\[ h: S/Q \vee G \rightarrow C \]

such that

\[ \pi(s) = \phi(s, h([s])). \quad (2) \]

Here $[s] \in S/Q \vee G$ is the equivalence class of $s$ under the combined action of $Q$ and $G$. Moreover any map $h$ generates an invariant $\pi$ through the relation (2).

The representation procedure quite naturally divides itself into two steps. The first involves the characterization of the range of invariant maps $\pi$, or, more precisely, description of the set $\mathcal{L} \subset S \times A$ within which the graph of $\pi$ must lie. The second step is then the actual construction of characterizing maps, a process which is frequently greatly simplified by the restrictive nature of $\mathcal{L}$. Theorem 36 states that it always is possible to construct direct representations as above and its proof outlines the procedure, while Theorem 32 describes an indirect charac-

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1 Frequently the action of $G$ on $A$ is trivial, so that this reduces to

\[ \pi(QsG) = Q\pi(s). \]