Approximate Solutions
of a Nonlinear Boundary Value Problem

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1. Introduction

The nonlinear boundary value problem $P_\varepsilon$ consisting of the differential equation

$\varepsilon y'' + F(t, y, y', \varepsilon) = 0$, \hspace{1cm} 0 \leq t \leq 1

and the boundary conditions

$y(0, \varepsilon) = \alpha(\varepsilon), \hspace{1cm} y(1, \varepsilon) = \beta(\varepsilon)

for the function $y = y(t, \varepsilon)$, and containing a small parameter $\varepsilon$, has been used repeatedly as a model of singular perturbation problems, in particular of boundary layer phenomena. Two features make the study of $P_\varepsilon$ significant. One concerns the existence of solutions. Coddington & Levinson [1] produced an example of a boundary value problem of the form of $P_\varepsilon$, with $\alpha$ and $\beta$ independent of $\varepsilon$, for which there exists a solution if $\varepsilon$ is sufficiently large but there exists no solution if $\varepsilon$ is positive and below a bound depending on $|\alpha - \beta|$, so that the solution collapses at a certain stage as $\varepsilon \to 0^+$. Thus, a verification of the continued existence of a solution of $P_\varepsilon$ as $\varepsilon \to 0$ is an important part of the investigation even in cases when a solution is known to exist for sufficiently large $\varepsilon$.

The second feature involves the behaviour of solutions as $\varepsilon \to 0$. If a solution $y = y(t, \varepsilon)$ exists for all sufficiently small $\varepsilon$, and if the limit $u(t) = \lim_{\varepsilon \to 0} y(t, \varepsilon)$ as $\varepsilon \to 0$ exists, it will be expected that $u$ satisfies

$F(t, u, u', 0) = 0$.

This indeed turns out to be the case under fairly liberal conditions. However, $u$ cannot be expected to satisfy both boundary conditions $u(0) = \alpha(0), u(1) = \beta(0)$; and the approach of $y$ to $u$ is non-uniform. The nature of this non-uniform approach gives valuable information about the singular perturbation problem $P_\varepsilon$ and deserves careful investigation.

Since the principal features of this problem depend considerably on the sign of $\varepsilon$, it will be assumed throughout this note that $\varepsilon$ is a positive parameter.

Some time ago [3, 4] I proved under fairly general and acceptable conditions the following result:

If (1.3) possesses a solution $u$ satisfying $u(1) = \beta(0)$, and if $|\alpha(0) - u(0)| < \mu_0$, where $\mu_0$ is a bound independent of $\varepsilon$, then $P_\varepsilon$ possesses a solution $y$ for all suf-
ficiently small $\varepsilon$, and for this solution

$$y = u + O(\varepsilon) + O(\varepsilon^{-\sigma(t)/\varepsilon}),$$
$$y' = u' + O(\varepsilon) + O(\varepsilon^{-1} e^{-\sigma(t)/\varepsilon})$$

holds uniformly for $0 \leq t \leq 1$. Here

$$\varphi(t) = \int_0^t F_{y'}(s, u(s), u'(s), 0) \, ds$$

is an increasing function of $t$ and $F_{y'} = \partial F/\partial y'$.

In the particular case when $F$ is a linear function of $y'$ (so that $F_{y'}$ is independent of $y'$) this result is included in older and more detailed results of Wasow [9]. Since the appearance of [9] and [4], further investigations were carried out. Harris [5], Macki [8], and Willett [10] replaced the single second order equation (1.1) by a system of two first order equations, studied more general boundary conditions than (1.2), and varied the conditions imposed upon $F$ in [4]. Lagerstrom [7] and J. D. Cole [2] explored the application of the general result described above to specific examples, in particular to the boundary value problem

$$\varepsilon y'' + y y' - y = 0, \quad 0 \leq t \leq 1,$$
$$y(0, \varepsilon) = \alpha, \quad y(1, \varepsilon) = \beta,$$

(1.4)

and compared the statement obtained for this boundary value problem from the general results of [4] with explicit constructions based on methods of practical applied mathematics.

More recently Willett [11] reconsidered $P_\varepsilon$ from a new point of view and made considerable further progress. He was able to relax somewhat the condition $F_{y'}(t, u(t), u'(t), 0) > 0$, seemingly eliminated the quantitative restriction on $|u(0) - \alpha(0)|$, and emphasized that his approach "includes the possibility of determining to any order of $\varepsilon$, the asymptotic nature of the solutions $y(t) = y(t, \varepsilon)$ as $\varepsilon \to 0^+$". Since the principal merit of [11] is not apparent, a word of explanation might be in order.

Willett does not construct approximations of higher order but assumes that functions are given which satisfy certain differential equations and boundary conditions to a stated degree of approximation. Neither does he eliminate the quantitative restriction on $|u(0) - \alpha(0)|$. In fact, he assumes the existence of an approximate solution $y^* = u + \varepsilon \Psi$ in his notation) for which $y^*(0, \varepsilon) - \alpha(\varepsilon)$ is small. A constructive determination of $y^*$ is not further discussed in [11], and whether or not $|u(0) - \alpha(0)| < \mu_0$ is necessary for a successful determination of $y^*$ may depend on the circumstances. The investigations in [7] and [2] show that even in the comparatively simple case of (1.4) a quantitative restriction on $u(0) - \alpha(0)$ is needed. This seems to indicate that the condition $|u(0) - \alpha(0)| < \mu_0$ cannot be eliminated without considerable restriction on the generality of the result formulated in [3, 4]. At the same time, in certain cases the quantitative condition is unnecessary, and in such cases Willett's results may indeed go much further than those of [9] and [4].