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SUBFORMULA SEMANTICS FOR STRONG NEGATION SYSTEMS

ABSTRACT. We present a semantics for strong negation systems on the basis of the subformula property of the sequent calculus. The new models, called subformula models, are constructed as a special class of canonical Kripke models for providing the way from the cut-elimination theorem to model-theoretic results. This semantics is more intuitive than the standard Kripke semantics for strong negation systems.

1. INTRODUCTION

Nelson [10] proposed 'constructible falsity' or 'strong negation' as the intuitionistic counterpart negation. A model-theoretic semantics for a strong negation system has been studied in detail by Thomason [13], Gurevich [6], and Akama [1] using Kripke models. Such models are modifications of Kripke models for intuitionistic logic so that both provability and refutability are represented in a parallel manner as in the original formulation of Kripke [9]. However the intuition behind the formalisms of these models remained unclear.

Kripke's original motivation for his semantics for intuitionistic logic is a model-theoretic analogue of the intuitionists' concept of proofs. It seems fruitful to consider a connection between particular proof systems and Kripke models. To develop a construction of Kripke models for particular proof methods produces philosophical justifications for Kripke's idea. However this aspect of research for Kripke models does not seem fruitful. In contrast, we find a clearer connection between Beth tableaux and Beth models.

The present paper aims at providing such a justification with reference to the so-called subformula property of the sequent calculus due to Gentzen [5]. The choice of this property is not essential, and a similar analysis may be obtained by means of such concepts as subordinate proofs of Fitch [4] and analytic tableaux of Smullyan [11].

Akama [2] develops a Gentzen-type sequent calculus and proves the cut-elimination theorem. We here assume the underlying proof theory to be this system. A similar semantics has already been proposed by
Ishimoto [7] for a propositional fragment of a strong negation system based on Schütte-type formulation; see Schütte [12]. Also, Akama [3] proposed a subformula semantics for intuitionistic predicate logic. This paper is self-contained, but the above mentioned papers on Kripke semantics for strong negation systems may be helpful to the reader.

2. SUBFORMULA AND SEQUENT CALCULUS

For details of sequent calculus and Kripke semantics for strong negation systems, the reader is referred to appropriate references. The strong negation system is designated as $S$, and the strong negation system with intuitionistic negation as $S^+$ though intuitionistic negation $\neg A$ can be defined as $A \supset \neg A$.

Now, we introduce several preliminaries. First, we define 'subformula' of a formula:

**Definition 2.1. (Subformula).** A subformula of a formula $A$ is defined inductively as follows:

1. if $A$ is a formula, then $A$ is a subformula of $A$.
2. If $A$ and $B$ are formulas, then the subformulas of $A$ and $B$ are subformulas of $A \land B$, $A \lor B$, and $A \supset B$.
3. If $A$ is a formula, the subformulas of $A$ are subformulas of $\neg A$ and $\neg A$ where $\neg$ denotes strong negation and $\neg$ intuitionistic negation.
4. If $x$ is a variable and $t$ is a term free for $x$ in a formula $A(x)$, the subformulas of $A(t)$ are subformulas of $\forall x A(x)$ and $\exists x A(x)$.

A sequent calculus for $S(S^+)$ is called $GS(GS^+)$. Now we will introduce a sequent $\Gamma \rightarrow \Delta$ of $GS$ where the Greek letters such as $\Gamma$ and $\Delta$ denote infinite (possibly empty) sets (not sequences) of formulas. The interpretation of a sequent in $GS$ is as usual. The expression of a pair $(\Gamma, \Delta)$ is introduced for characterizing a sequent. The pair $(\Gamma, \Delta)$ is dual consistent if $\Gamma \rightarrow \Delta$ is not provable in $S$. $S(A)$ denotes a set of all the subformulas of the formula $A$. A dual consistent pair $(\Gamma, \Delta)$ is complete if $\Gamma \cup \Delta = S(F)$. If $(\Gamma, \Delta)$ is complete then $\Gamma \cap \Delta = \emptyset$. If we confine our