On the Linear Instability of the Hall–Stewartson Vortex

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Abstract. It is demonstrated that the Hall–Stewartson leading-edge vortex is linearly unstable to viscous perturbations of the center-mode type. Center modes are found to occur in two regions of Reynolds-number–wave-number space, in limits in which the axial wave number is large. The appropriate center-mode equations in these neighborhoods are established, and it emerges that the two sets are identical. The single system of equations, which depends on the azimuthal wave number \( m \) and a distance parameter \( \sigma \) only, is solved numerically for various values of \( m \) and \( \sigma \). Highly unstable modes are found for large positive \( \sigma \), and the results are shown to be in good agreement with proposed asymptotic expansions when \( |\sigma| \gg 1 \). To lowest order, unstable modes have phase surfaces that rotate with the fluid: in addition constant phase surfaces propagate upstream but the group velocity is directed downstream. The growth rate of the instability increases faster than Reynolds number to the quarter power. This, together with the finding that the length scale of the unstable modes found goes to zero as the Reynolds number tends to infinity, makes this instability an unusual one.

1. Introduction

It is more than three decades since Brown and Michael (1954) studied the effect of leading-edge separation on the lift of a slender delta wing at incidence by modeling each vortex that springs from the leading edge by a single line element. Pairs of such vortices, robust columns of fluid before disruption takes place, lie above the wing, inboard of the leading edges, and are beautifully depicted in the now classic experiments of Lambourne and Bryer (1961) and of Werlé (1960). The vortices are present over most of the operating speed range and incidence as long as the leading edge remains subsonic, and the very low pressures that occur in the core are associated with the production of lift. During the early period of development of European supersonic civilian transport considerable effort was made to analyze models, some of which preceded that of Brown and Michael, that would lead to a description of the flow within the vortex itself. Roy (1952), Legendre (1953), and a particularly successful study by Mangler and Smith (1959), modeled the vortex by a continuous sheet separated
from the leading edge between the turns of which the flow is irrotational. Comprehensive reviews of these and related models are given by Smith (1982, 1984).

In most of the aforementioned inviscid studies the flow is assumed to be incompressible and conical (a flow is said to be conical if the velocities and pressure are independent of the distance along rays through the origin) although certainly not necessarily axisymmetric. However, the general assumption made that is likely to be most at variance with the observations is that of slender-body theory. The experiments of Earnshaw (1961), for example, show axial velocities in the core that differ considerably from those in the main stream, indicating that linearization is not justified. An alternative model which obviates the difficulty of slender-body theory is that of Hall (1961). Hall presented an exact rotational conical axisymmetric solution of the incompressible Euler equations that predicts logarithmically infinite velocities and correspondingly low pressures in the core. Hall also gave a viscous correction that removed the singularities at the symmetry axis. A subsequent paper by Stewartson and Hall (1963) eliminated inconsistencies in Hall's treatment of the viscous core by means of a consistent asymptotic analysis valid in the limit of large Reynolds number. Comparison of the theory with the experiments of Earnshaw showed good quantitative agreement with the observations, although the theory somewhat overestimated the axial and circumferential velocities and consequent pressure drop. A complementary inviscid model that does not employ slender-body assumptions is that of Mangler and Weber (1967). These authors represent the vortex by the continuous turns of a vortex sheet, the flow between the turns being irrotational. A point of note is that as the center is approached and the sheet becomes tightly wound the solution asymptotes to the rotational Hall's vortex. A continuation to all space of such a conical flow over a sector (a delta wing for example) cannot, as demonstrated by Legendre (1956), be made without the occurrence of singularities in the flow. Thus an embedding of these vortices in any external flow must be made by asymptotic or empirical means.

Our concern here is with the stability of the Stewartson–Hall vortex. This flow has not been the subject of as much subsequent study as the development of potential slender-body models, possibly because it is not clear how best to relate it to a realistic outer flow. Nevertheless, it has an advantage over the slender-body models in that it predicts the considerable variations of velocities and pressure across the core.

We find that the vortex is linearly unstable to nonaxisymmetric perturbations that are concentrated in the neighborhood of the axis, the so-called "center-modes." The theoretical overprediction of velocity on the axis evident in Hall's (1961) comparison of his result and the measurements of Earnshaw may therefore be due to mixing of lower momentum fluid from the outer part of the core with that in the immediate neighborhood of the axis.

It is conventional wisdom that viscous effects damp disturbances of very short wavelengths. Here we will find that this is not the case. Unstable center modes with azimuthal wave number m have asymptotic dimensional growth rates given by

$$3 \frac{\pi^{1/4} m^{5/6} \sigma^{1/3} R^{1/4}}{32} \log R$$

as the Reynolds number R, based upon a measure of the speed at the bounding cone and distance from the cone vertex, tends to infinity. Here \( \sigma \) is a parameter, independent of R, that serves to define two strips in the axial wave-number–Reynolds-number plane, each of which contains a neutral curve. Far from being damped by viscosity, these modes have growth rates that tend to infinity as the Reynolds number tends to infinity, which is also unusual. Thus, the instability is novel in two respects. The increasing rates of strain in the Stewartson–Hall vortex, as Reynolds number increases, leads to an increase of the maximum possible rate of production of disturbance energy, and this accounts for the unusual stability characteristics of this flow.

Like the "ring modes" (unstable modes with perturbations concentrated about a cylindrical surface of nonzero radius) discussed by Leibovich and Stewartson (1983), the unstable center modes that we find here exist only when surfaces of constant phase have positive screw sense. Constant phase surfaces rotate about the symmetry axis in the same direction as the fluid in the basic flow rotates, but do so in a retrograde fashion, rotating more slowly than the fluid.

Viscous center modes were first encountered, as the Reynolds number increased, by Cotton and Salwen (1981) in their computations of swirling Poiseuille flow. The equations satisfied by these center modes were analyzed by Stewartson et al. (1988) (a paper hereinafter referred to as SNB). In these