On the Global Stability of the Logistic Age-Dependent Population Growth

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Abstract. We study an age-dependent population equation with a nonlinear death rate of "logistic" type. The global asymptotic stability of the null solution is investigated when \( R(0) < 1 \). If \( R(0) > 1 \) we get the existence of a nontrivial steady state that becomes asymptotically stable itself, while the null solution is unstable. The rate of decay is estimated.

Key words: Logistic growth – Age dependent dynamics – Global stability

In mathematical demography a quite simple but interesting model is the so-called "logistic growth" model due to Verhulst [17]. Let us denote by \( P(t) \) the total population at time \( t \), the dynamics are described by means of the following ordinary differential equation

\[
\frac{dP}{dt} = P(t)[(\beta - \mu) - \gamma P(t)], \quad P = \frac{dP}{dt}, \tag{1}
\]

where \( \beta \) is the birth rate, \( \mu \) is the intrinsic death rate, and \( \gamma \) is a constant depending on the self-limitation phenomena.

This model is characterized by a linear "renewal law" and by a "death rate" of the form \( \mu + \gamma P \) linear in the total population size. Using these simple assumptions an age-dependent version of the above model can be obtained (see [18]).

Denote by \( u(a, t) \) the density of population per unit age at time \( t \). The total population size is

\[
P(t) = \int_{0}^{A} u(a, t) \, da, \tag{2}
\]

where \([0, A]\) is the maximum life span. The related age-dependent model can be described using the following equations of hyperbolic type

\[
u_{a} + u_{t} + \mu(a)u + \gamma P(t)u = 0, \quad \gamma > 0,
\]

\[
u(0, t) = \int_{0}^{A} \beta(a)u(a, t) \, da, \quad P(t) = \int_{0}^{A} u(a, t) \, da,
\]

\[
u(a, 0) = \phi(a), \quad a \in [0, A], \quad t \geq 0, \tag{3}
\]

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where \( \phi \in C^1[0, A], \phi \geq 0, \phi(A) = 0, \) and
\[
\phi(0) = \int_0^A \beta(a)\phi(a) \, da.
\]
For our purposes it is sufficient to assume
\[
\beta \in C[0, A], \quad \beta > 0 \text{ in } (0, A), \quad \beta(0) = \beta(A) = 0,
\]
\[
\mu \in C[0, A], \quad \mu > 0, \quad \int_0^A \mu(a) \, da = +\infty.
\]
The quantity
\[
p(a) = \exp \left[ -\int_0^a \mu(x) \, dx \right]
\]
represents the probability to survive till age \( a, \) and
\[
R(P) = \int_0^A \beta(a)p(a)\exp[-\gamma P] \, da, \quad P \geq 0,
\]
the "net reproduction rate".

Existence and uniqueness of positive solutions of this kind of problem have been widely studied. The most important references are Gurtin and MacCamy [4], [5], Webb [14], [15], [16] and Di Blasio [2]. Stability results concerning the local theory can be found in Gurtin and MacCamy [5] and in [11]. Some stability results for models where the nonlinearity is in the birth rate can be found in Rorres [12], [13], Lamberti and Vernole [7] and Di Blasio et al. [3]. To the author’s knowledge their techniques cannot be applied to models having a nonlinear death law, as in the present case.

The stability properties both in the above mentioned papers and in our model are given with respect to the quantity \( R(0) \). This feature is not surprising since \( R(0) \) can be considered as the "number of sons" generated by a single person during his life, when the dynamics are observed without taking into account the self-limitation phenomena.

Our first result is concerned with the global attractivity of the null solution when \( R(0) < 1 \) (we point out that \( R(0) < 1 \) is equivalent to saying \( \beta < \mu \) for the logistic Eq. (1)).

**Proposition 1.** If \( R(0) < 1 \), let us denote by \( p_0 \) the unique real root of (see [6])
\[
\int_0^A \beta(a)p(a)e^{-\lambda a} \, da = 1,
\]
then \( p_0 < 0 \) and
\[
u(a, t) = 0(e^{p_0t}), \quad P(t) = 0(e^{p_0t})
\]
uniformly in \( a \in [0, A] \), as \( t \to +\infty \).

**Proof.** With the transformation
\[
w(a, t) = \exp \left[ \gamma \int_0^t P(s) \, ds \right] \nu(a, t)
\]
(6)