Bounds for the Ratios
of the First Three Membrane Eigenvalues

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1. Introduction. Let $D$ be a bounded domain in the $x, y$-plane with sufficiently smooth boundary $C$. We consider the membrane eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad D,$$
$$u = 0 \quad \text{on} \quad C.$$

There are infinitely many positive eigenvalues $\lambda_i$:

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots.$$

The corresponding normalized eigenfunctions are

$$u_1, u_2, u_3, \ldots.$$

In [1] Payne, Pólya & Weinberger have shown that

$$\lambda_{n+1} \leq \lambda_n + \frac{2}{n} (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \leq 3 \lambda_n. \quad (1)$$

For $n = 1$ this implies $\lambda_2/\lambda_1 \leq 3$. Moreover they conjectured that $\lambda_{n+1}/\lambda_n \leq \tilde{\lambda}_2/\tilde{\lambda}_1 = 2.539\ldots$ where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are the first two eigenvalues of a circular domain. In [1] it is also proved that for any domain $D$

$$\lambda_2 + \lambda_3 \leq 6 \lambda_1. \quad (1')$$

In this note we shall sharpen this inequality to

$$\lambda_2 + \lambda_3 \leq 5 \lambda_1 + \lambda_2^2/\lambda_2. \quad (2)$$

In conjunction with (1) (viz $\lambda_3 \leq 2 \lambda_2 + \lambda_2$) and $\lambda_3 \leq \lambda_2$ this will be shown to lead to

$$\frac{\lambda_2}{\lambda_1} \leq \frac{5 + \sqrt{33}}{4} = 2.686\ldots, \quad (3)$$

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 3 + \sqrt{7} = 5.646\ldots, \quad (4)$$

$$\frac{\lambda_3}{\lambda_1} \leq \frac{7 + 2\sqrt{7}}{3} = 4.097\ldots. \quad (5)$$
Our method consists of introducing a parameter \( \alpha \) in the method of [1] (the result of [1] is obtained upon specialization \( \alpha = 1 \)) and taking its value such that the final result is optimal.

We restrict ourselves in this paper to \( \lambda_n \) with \( n = 1, 2, 3 \); for further values of \( n \) the method becomes somewhat complicated.

2. Proof of (2). In the following we shall use the notation

\[
\int f = \int \int_D f(x, y) \, dx \, dy, \quad u_{ix} = \frac{\partial}{\partial x} u_i
\]

etc. We choose rectangular coordinates such that

\[
\int x u_i^2 = 0, \quad \int y u_i^2 = 0 \quad \text{and} \quad \int x u_1 u_2 = 0.
\]

This is always possible by taking as origin the center of gravity of \( D \) with mass-distribution \( u_i^2 \). After that we can make \( \int x u_1 u_2 \) zero by a rotation of the coordinate system.

It is well known that \( u_1 \) does not change sign in \( D \); hence we may choose \( u_1 > 0 \) in \( D \).

We shall use the following minimum principle for the eigenvalues. Let \( \Sigma_1 \) denote the set of all sufficiently smooth functions which are defined on \( D + C \) and which are zero on \( C \), and for \( k > 1 \) let \( \Sigma_k \) consist of the functions \( \varphi \in \Sigma_1 \) for which \( \int u_i \varphi = 0, \, i = 1, 2, \ldots, k - 1 \). Then

\[
\lambda_n = \min_{\varphi \in \Sigma_n} \frac{\int (\varphi_x^2 + \varphi_y^2)}{\int \varphi^2} = \int (u_{inx}^2 + u_{inyy}^2), \quad n = 1, 2, \ldots.
\]

As in [1] we take the trial functions \( \varphi_1 = x u_1 \) and \( \varphi_2 = y u_1 \); by (6) it is clear that \( \varphi_1 \in \Sigma_1 \) and \( \varphi_2 \in \Sigma_2 \). We thus obtain the inequalities

\[
\lambda_2 \leq \int [u_1^2 \varphi + 2 x u_1 u_{1x} + x^2 (u_{1xx} + u_{1yy})] / \int u_{1xx}^2, \\
\lambda_1 \leq \int [u_1^2 \varphi + 2 y u_1 u_{1y} + y^2 (u_{1yy} + u_{1xx})] / \int u_{1yy}^2.
\]

By partial integration we find

\[
2 \int x u_1 u_{1x} = - \int u_1^2 = - 1 = 2 \int y u_1 u_{1y}
\]

and also

\[
\int x^2 u_{1x}^2 = - \int u_1 (2 x u_{1x} + x^2 u_{1xx}), \\
\int x^2 u_{1y}^2 = - \int u_1 x^2 u_{1yy}.
\]

Addition of these two equalities gives (using (9))

\[
\int x^2 (u_{1x}^2 + u_{1y}^2) = - 2 \int x u_1 u_{1x} - \int x^2 (u_{1xx} + u_{1yy}) u_1 = 1 + \lambda_1 \int x^2 u_1^2.
\]

In the same way we find

\[
\int y^2 (u_{1x}^2 + u_{1y}^2) = 1 + \lambda_1 \int y^2 u_1^2.
\]

We substitute (9), (10) and (11) into (8), and we arrive at the inequalities

\[
\lambda_2 \leq \lambda_1 + (\int x^2 u_1^2)^{-1}, \quad \lambda_2 \leq \lambda_1 + (\int y^2 u_1^2)^{-1};
\]

or, after addition,

\[
\lambda_2 + \lambda_2 \leq 2 \lambda_1 + (\int x^2 u_1^2)^{-1} + (\int y^2 u_1^2)^{-1}.
\]