On the Limit Cycles of $\dot{x} + \mu \sin \dot{x} + x = 0$

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Introduction

This article is concerned with the existence of limit cycles of the differential equation

$$\dot{x} + \mu f(x) + x = 0 \quad (1)$$

where in particular

$$f(x) = \sin x. \quad (2)$$

An equation of the form (1) with

$$f(x) = -x + \frac{1}{3} \dot{x}^3 \quad (3)$$

is known as van der Pol's equation ([3] p. 284). For the latter one can show that (1) has precisely one limit cycle. When $f(x)$ is an odd polynomial of degree $2n+1$ and with $2n+1$ simple real zeros, one can show that (1) has precisely $n$ limit cycles for sufficiently small $\mu$. It is plausible therefore that when $f(x) = \sin x$, (1) should have an infinite number of limit cycles.

It is indeed the case that for (1), (2) an infinite number of limit cycles exists, for $\mu$ sufficiently small. But the classical perturbation technique that is customarily employed breaks down. Full treatments of the classical procedure may be found in CESARI [1] (p. 136 ff.), CODDINGTON & LEVINSON [2] (p. 348 ff.), and HOCHSTADT [3] (p. 279 ff.). The reason for this breakdown is the following. If one seeks a solution of (1) in the form

$$x = A \cos \omega(t) t + \mu x_1 + \cdots \quad (4)$$

where $A$ is an amplitude and the frequency $\omega$ is a suitable function of $\mu$, one finds that $A$ must satisfy the secular equation

$$\int_0^{2\pi} f(A \sin \varphi) \sin \varphi \, d\varphi = 0. \quad (5)$$

For each zero of (5) one can determine a $\bar{\mu} > 0$ such that (1) has a limit cycle, which is represented by (4) for all $|\mu| < \bar{\mu}$. If (5) has a finite number of zeros, one can establish a positive bound $\bar{\mu}$, so that for $|\mu| < \bar{\mu}$ (1) has a limit cycle near all zeros of (5). When (5) has an infinite number of zeros, such a positive bound may not exist.

For (3), (5) reduces to

$$\int_0^{2\pi} (-A \sin \varphi + \frac{1}{3} A^3 \sin^3 \varphi) \sin \varphi \, d\varphi = -\frac{1}{2} A + \frac{1}{8} A^3 = 0.$$
$A=0$ yields $x=0$, and $A = \pm 2$ yields one and the same limit cycle. For the case (2) we find

$$\int_0^{2\pi} \sin(A \sin \varphi) \sin \varphi \, d\varphi = 2\pi J_1(A) = 0$$

(6)

where $J_1(A)$ is the first order Bessel function. It is well known that it has an infinite number of real, simple zeros. It will be shown that there exists a $\tilde{\mu} > 0$ such that for $|\mu| < \tilde{\mu}$ (1) has a limit cycle near each zero of $J_1(A)$. ECKWEILER [4, 5] observed this, but felt that the extension of the proof from the case with a finite number of zeros to that with an infinite number was merely a matter of detail. In fact an extension following the classical technique may not be possible. The discussion of this article is based on topological techniques and the proof of the existence of an infinite number of limit cycles is based on the Brouwer fixed point theorem. The results of this article appear to be the first regarding an equation of type (1) with an infinite number of limit cycles.

**Outline of Proof**

In studying the differential equation

$$\ddot{x} + \mu \sin \dot{x} + x = 0$$

(7)

it will prove to be convenient to introduce polar coordinates. We let

$$x = r \cos \theta,$$

$$\dot{x} = r \sin \theta,$$

so that (7) is replaced by the system

$$\dot{r} \cos \theta - r \sin \theta \dot{\theta} = r \sin \theta,$$

$$\dot{r} \sin \theta + r \cos \theta \dot{\theta} = - r \cos \theta - \mu \sin (r \sin \theta).$$

(8)

Solving for $\dot{r}$ and $\dot{\theta}$, and eliminating the independent variable, one obtains

$$\frac{dr}{d\theta} = \mu \sin \theta \sin (r \sin \theta)$$

$$1 + \mu \cos \theta \sin (r \underline{\sin} \theta)$$

(9)

(9) is more convenient to deal with than (7). First of all (9) is a first order equation. Secondly, the period of any limit cycle in (7) will be a function of $\mu$. A periodic solution of (9), which is equivalent to a limit cycle of (7), will have period $2\pi$ in the angular variable $\theta$. (9) can be rewritten as an integral equation

$$r(\theta, \mu) = A + \int_0^\theta \frac{\mu \sin \varphi \sin (r \sin \varphi)}{1 + \mu \cos \varphi \sin (r \underline{\sin} \varphi)} \, d\varphi.$$  

(10)

In order for $r$ to have period $2\pi$ we require that

$$r(2\pi, \mu) = A,$$